Parameter-free preconditioning for nearly-incompressible linear elasticity

James H. Adler [a], Xiaozhe Hu [a], Yuwen Li [b], Ludmil T. Zikatanov [c,*]

[a] *Department of Mathematics, Tufts University, Medford, MA 02155, USA*
[b] *School of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310058, China*
[c] *Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA*

ARTICLE INFO

ABSTRACT

It is well known that via the augmented Lagrangian method, one can solve Stokes' system by solving the nearly incompressible linear elasticity equation. In this paper, we show that the converse holds, and approximate the inverse of the linear elasticity operator with a convex linear combination of parameter-free operators. In such a way, we construct a uniform preconditioner for linear elasticity for all values of the Lamé parameter $\lambda \in [0, \infty)$. Numerical results confirm that by using inf-sup stable finite-element spaces for the solution of Stokes' equations, the proposed preconditioner is robust in λ.

1. Introduction

The main focus of this work is on developing and analyzing an effective preconditioning operator for the primal formulation of linear elasticity, particularly in the incompressible limit. For a body force, $\tilde{\mathbf{f}}$, acting on an isotropic elastic material, we model the displacement of the deformable media, $\mathbf{u}$, via the governing equation,

$$\text{div}\left(2\mu\varepsilon(\mathbf{u}) + \tilde{\lambda}\text{tr}(\varepsilon(\mathbf{u}))I\right) = \tilde{\mathbf{f}}. \tag{1.1}$$

Here, μ, $\tilde{\lambda}$ are Lamé parameters, tr is the trace operation for tensors, I is the identity tensor, and the strain tensor, $\varepsilon(\mathbf{u})$, is given by

$$\varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right).$$

In terms of Poisson ratio, ν, and Young's modulus, E, the Lamé constants are expressed as

$$\mu = \frac{E}{2 + 2\nu}, \quad \tilde{\lambda} = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}, \quad 0 \leq \nu < \frac{1}{2}.$$

The linearly elastic material becomes nearly incompressible when $\nu \to \frac{1}{2}^-$ and $\tilde{\lambda} \to \infty$ (cf. [1]).

In the incompressible limit, traditional finite-element and finite-difference schemes suffer from volumetric/Poisson locking. This locking phenomenon is due to the poor representation of the divergence-free vector fields [2] in the underlying space. It is therefore not surprising that locking-free numerical schemes of linear elasticity are related to discretization methods for Stokes' equations. A quick look at the paper by Bramble [3] reveals that the Stokes' inf-sup condition implies the fundamental 2nd Korn's inequality in elasticity. Such ideas have led to the development of stable and accurate numerical methods for Stokes' equation via the *augmented Lagrangian* formulation, see, e.g., [4–6]. Analysis of the corresponding iterative solution techniques for the resulting linear systems is studied in [7], and the approach has been successful in a variety of related applications [8–10].

Conversely, results from solving Stokes' equations can be used for developing schemes for nearly incompressible linear elasticity. For instance, in [1], Braess introduces an auxiliary variable, $p = \text{div }\mathbf{u}$, and uses the stability of a perturbed Stokes' problem to derive *a priori* error estimates which are uniform with respect to $\tilde{\lambda}$. Further works by Schöberl [11] and Carstensen [12] utilize similar ideas to analyze and show robustness of multigrid solvers and reliability and efficiency of *a posteriori* error estimation for finite-element discretizations of linear elasticity. We note that, as shown in [13], the results derived here, combined with the operator preconditioning framework in [14,15], can be utilized to design novel *a posteriori* error estimators for nearly incompressible linear elasticity.

Based on this notion, the main contribution of this work is to use stable discretizations of Stokes' equations to develop a preconditioner for linear elasticity that, unlike most others, is provably robust and performs uniformly well for all values of $\tilde{\lambda} \in [0, \infty)$. The main ingredients in the construction are: (1) the action of the inverse of a standard, parameter-free, elliptic operator; and (2) computing an H^1-type orthogonal projection onto the space of (discrete) divergence-free vector fields. We note here that computing the projection requires solving an-

* Corresponding author.
E-mail address: ludmil@psu.edu (L.T. Zikatanov).

other parameter-free (discrete) Stokes' problem. The underlying idea comes from a simple observation concerning linear elasticity with periodic boundary conditions, under which our preconditioner reduces to the exact inverse of the linear elasticity operator. In general, the preconditioner is not the exact inverse, but is very close to it. Such claims are validated by numerical tests showing that the preconditioned linear system corresponding to discretizations of (1.1) has uniformly bounded condition numbers.

This rest of the paper is organized as follows. Section 2 sets up the bilinear forms and notation used throughout the paper. In Section 3, an inf-sup condition and Korn inequality is established to help build a parameter-free preconditioner. Next, the spectral equivalence result that yields the robust preconditioner is given in Section 4. The case of periodic boundary conditions is also considered here. Finally, numerical results confirming the theory are shown in Section 5, with concluding remarks given in Section 6.

2. Preliminaries and notation

Let $\Omega \subset \mathbb{R}^d$ with $d \in \{2,3\}$ be a bounded polyhedron with Lipschitz boundary. Let $(\cdot, \cdot)$ denote the $L^2(\Omega)$ inner product, $\|\cdot\|$ the $L^2(\Omega)$ norm, $Q = L^2(\Omega)$, and $V \subset [H^1(\Omega)]^d$ be a Hilbert space. By $\langle \cdot, \cdot \rangle$ we denote the duality pairing between V and its dual V' or Q and its dual Q'. The boundary of Ω is $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$, where Γ_D is a closed set with respect to Γ with a nonzero $(d-1)$ dimensional measure. Further, we denote by $[H^1_D(\Omega)]^d \subset [H^1(\Omega)]^d$ the space of vector-valued functions on Ω with vanishing traces on Γ_D. Often, $V = [H^1_D(\Omega)]^d$, however, we also consider examples with periodic boundary conditions on the unit cube in $\mathbb{R}^d$ and thus, V and Q will be modified accordingly.

The variational problem of (1.1) is to find $\mathbf{u} \in V$ such that

$$a_\lambda(\mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) + \lambda b(\mathbf{v}, \operatorname{div} \mathbf{u}) = \langle \mathbf{f}, \mathbf{v} \rangle, \tag{2.1}$$

for all $\mathbf{v} \in V$, where $\mathbf{f} := \tilde{\mathbf{f}}/(2\mu)$, $\lambda := \tilde{\lambda}/(2\mu)$, and

$$a(\mathbf{u}, \mathbf{v}) = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})),$$

$$b(\mathbf{v}, q) = (\operatorname{div} \mathbf{v}, q).$$

Note that we divide the original equation by 2μ and obtain a modified parameter

$$\lambda = \frac{\nu}{1 - 2\nu}, \quad 0 \le \nu < \frac{1}{2}.$$

The bilinear forms in (2.1) define operators $A_\lambda : V \to V'$, and $B : V \to Q'$ by

$$\langle A_\lambda \mathbf{u}, \mathbf{v} \rangle := a_\lambda(\mathbf{u}, \mathbf{v}),$$
$$\langle B\mathbf{v}, q \rangle := b(\mathbf{v}, q) = (\operatorname{div} \mathbf{v}, q). \tag{2.2}$$

As $\lambda = 0$ is a special case, we define $A := A_0$ with

$$\langle A\mathbf{u}, \mathbf{v} \rangle = a(\mathbf{u}, \mathbf{v}) := (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})). \tag{2.3}$$

For any two operators, X and Y mapping a space V to its dual V', we write $X \lesssim Y$ when $\langle Xv, v \rangle \le C \langle Yv, v \rangle$ holds for any $v \in V$ with a generic constant C depending on Ω and independent of λ and μ. By $X \approx Y$, we denote $X \lesssim Y$ *and* $Y \lesssim X$. Then, the goal of this paper is to show that

$$A_\lambda^{-1} \approx \frac{\lambda}{\lambda+1} P A^{-1} + \frac{1}{\lambda+1} A^{-1} =: M_\lambda, \tag{2.4}$$

where P is the $a_\lambda(\cdot, \cdot)$-orthogonal projection onto the space of divergence-free vector fields. Clearly P can be implemented by solving Stokes' equations.

3. Brezzi's inf-sup condition and Korn's inequality

In order to develop a robust preconditioner, we consider some properties related to the inf-sup condition on $V \times Q$ (see e.g. [16–18]):

$$\inf_{q \in Q} \sup_{\mathbf{v} \in V} \frac{(\operatorname{div} \mathbf{v}, q)}{\|\nabla \mathbf{v}\| \|q\|} \ge \tilde{\beta} > 0. \tag{3.1}$$

As is shown in [3], (3.1) is equivalent to the following inequality due to Nečas [19]:

$$\|\mathbf{u}\| \lesssim \left(\|\mathbf{u}\|^2_{H^{-1}(\Omega)} + \sum_{j=1}^{d} \left\| \frac{\partial \mathbf{u}}{\partial x_j} \right\|^2_{H^{-1}(\Omega)} \right)^{1/2} \tag{3.2}$$

In addition, [3] shows that (3.2) implies Korn's inequality:

$$\|\nabla \mathbf{u}\| \lesssim \|\mathbf{u}\| + \|\varepsilon(\mathbf{u})\|, \quad \forall \mathbf{u} \in [H^1(\Omega)]^d. \tag{3.3}$$

The classical Korn's inequality is found in [20,21]. The proof of this inequality is simple under Dirichlet boundary conditions. The situation is much more complicated in the case of traction conditions on part of the boundary. We refer to Kondratiev and Oleinik in [22,23], Duvaut and Lions [24], Nitsche [25], and Bramble [3] for proofs of various types of Korn's inequalities. An important consequence of (3.3) is the following lemma which shows the coercivity of $a(\cdot, \cdot)$. For completeness, we include a proof of this well known result following [26] (see Appendix A).

Lemma 3.1 (*[26], p. 27*). *Let $\mathfrak{R}$ be the space of rigid body motions,*

$$\mathfrak{R} = \left\{ \mathbf{c} + \mathfrak{m}\mathbf{x} \mid \mathbf{c} \in \mathbb{R}^d, \quad \mathfrak{m} \in \mathfrak{so}(d) \right\},$$

where $\mathbf{x}$ is the position vector in $\mathbb{R}^d$ and $\mathfrak{so}(d)$ is the algebra of the real and anti-symmetric $d \times d$ matrices. Then it holds that

$$\|\nabla \mathbf{u}\| \lesssim \|\varepsilon(\mathbf{u})\|, \quad \forall \mathbf{u} \in [H^1_D(\Omega)]^d \cup \left([H^1(\Omega)]^d \cap \mathfrak{R}^{\perp_{L^2}} \right). \tag{3.4}$$

We exploit the fact that the linear elasticity problem for large λ can be viewed as a penalty formulation of a constrained minimization problem (see, e.g. [27,2,3,19]). We introduce the subspace of divergence-free functions,

$$W := \operatorname{Ker}(B) = \left\{ \mathbf{v} \in V \mid B\mathbf{v} = \operatorname{div} \mathbf{v} = 0 \right\}.$$

As $B = \operatorname{div}$ is a continuous operator, its kernel is a closed subspace of V. Then, (3.4) and the equivalence between (3.1) and (3.2) imply that $a(\cdot, \cdot) = (\varepsilon(\cdot), \varepsilon(\cdot))$ is an inner product on V with corresponding norm equivalent to the $\|\cdot\|_{[H^1(\Omega)]^d}$ norm. This yields the following inf-sup conditions, equivalent to (3.1):

$$\inf_{q \in Q} \sup_{\mathbf{v} \in V} \frac{(\operatorname{div} \mathbf{v}, q)}{\|\varepsilon(\mathbf{v})\| \|q\|} = \inf_{\mathbf{v} \in W^\perp} \sup_{q \in Q} \frac{(\operatorname{div} \mathbf{v}, q)}{\|\varepsilon(\mathbf{v})\| \|q\|} \ge \beta > 0. \tag{3.5}$$

Here, the orthogonality in $W^\perp$ is in terms of the inner product $a(\cdot, \cdot)$. For the proof of the equivalence between the two conditions in (3.5), we refer to Girault and Raviart [27, Lemma 4.1].

Next, we define $P : V \to W$ to be the orthogonal projection onto W with respect to $a(\cdot, \cdot)$. In other words, for $\mathbf{v} \in V$ the projection $P\mathbf{v} \in W$ is the unique solution to

$$a(P\mathbf{v}, \mathbf{w}) = a(\mathbf{v}, \mathbf{w}), \quad \forall \mathbf{w} \in W. \tag{3.6}$$

It is immediate to see that $\mathbf{v}_0 = P\mathbf{v}$ solves the Stokes' equation:
Find $(\mathbf{v}_0, p) \in V \times Q$ such that

$$
\begin{aligned}
a(\mathbf{v}_0, \mathbf{w}) + b(\mathbf{w}, p) &= a(\mathbf{v}, \mathbf{w}), & \forall \mathbf{w} \in V, \\
b(\mathbf{v}_0, q) &= 0, & \forall q \in Q,
\end{aligned} \tag{3.7}
$$

where the "pressure" p serves as a Lagrange multiplier for the divergence free constraint.

4. Spectral equivalence and a robust preconditioner

In this section, we use the aforementioned relationship between the inf-sup condition and Korn's inequality to develop a robust preconditioner, M_λ, for the linear elasticity equations. We start by proving the

spectral equivalence (2.4) between M_λ and the inverse of the linear elasticity operator, A_λ.

Theorem 4.1. *If $M_\lambda : V' \to V$ is defined by*

$$M_\lambda = \frac{\lambda}{1+\lambda} P A^{-1} + \frac{1}{\lambda+1} A^{-1},$$

where A_λ, A, and P are as in (2.2), (2.3) and (3.6). Then,

$$\langle \mathbf{g}, M_\lambda \mathbf{g} \rangle \approx \langle \mathbf{g}, A_\lambda^{-1} \mathbf{g} \rangle \quad \forall \mathbf{g} \in V'. \tag{4.1}$$

Proof. To show the spectral equivalence, it suffices to prove that for any $\lambda \in [0, \infty)$ and $\mathbf{v} \in V$,

$$\langle A_\lambda \mathbf{v}, \mathbf{v} \rangle \lesssim \langle M_\lambda^{-1} \mathbf{v}, \mathbf{v} \rangle \lesssim \langle A_\lambda \mathbf{v}, \mathbf{v} \rangle.$$

It follows from $P^2 = P$ that

$$(I + tP)^{-1} = I - \frac{t}{t+1} P \quad \forall t \in \mathbb{R} \setminus \{-1\}. \tag{4.2}$$

Using this fact, we obtain

$$\begin{aligned}
M_\lambda^{-1} &= \left(\frac{1}{\lambda+1} (I + \lambda P) A^{-1} \right)^{-1} \\
&= A((\lambda+1)I - \lambda P) \\
&= A + \lambda A(I - P).
\end{aligned} \tag{4.3}$$

By comparing (4.3) with $\langle A_\lambda \mathbf{v}, \mathbf{v} \rangle = \langle A\mathbf{v}, \mathbf{v} \rangle + \lambda \|\operatorname{div} \mathbf{v}\|^2$, it remains to show that

$$\begin{aligned}
\|\operatorname{div} \mathbf{v}\|^2 &\approx \langle A(I - P)\mathbf{v}, \mathbf{v} \rangle \\
&= a(\mathbf{v} - P\mathbf{v}, \mathbf{v} - P\mathbf{v}) = \|\varepsilon(\mathbf{v} - P\mathbf{v})\|^2.
\end{aligned} \tag{4.4}$$

The lower bound in (4.4) directly follows from

$$\begin{aligned}
\|\operatorname{div} \mathbf{v}\| &= \sup_{q \in Q, \|q\|=1} (\operatorname{div}\mathbf{v}, q) = \sup_{q \in Q, \|q\|=1} (\operatorname{div}(\mathbf{v} - P\mathbf{v}), q) \\
&\le \|\operatorname{tr}[\varepsilon(\mathbf{v} - P\mathbf{v})]\| \le \sqrt{d} \, \|\varepsilon(\mathbf{v} - P\mathbf{v})\|.
\end{aligned} \tag{4.5}$$

The upper bound is just a restatement of (3.5) because $(I - P)\mathbf{v} \in W^\perp$. In fact, we have

$$\begin{aligned}
\beta \|\varepsilon(\mathbf{v} - P\mathbf{v})\| &\le \sup_{q \in Q, \|q\|=1} \left(\operatorname{div}(\mathbf{v} - P\mathbf{v}), q \right) \\
&= \sup_{q \in Q, \|q\|=1} \left(\operatorname{div}\mathbf{v}, q \right) = \|\operatorname{div} \mathbf{v}\|,
\end{aligned} \tag{4.6}$$

and this completes the proof of (4.4) and, hence, (4.1). $\square$

4.1. Periodic boundary conditions

To expose the main idea for the preconditioner and motivate how to tackle a more general case, we investigate the case of periodic boundary conditions. Here, the spectral equivalence is in fact an equality ("$\approx$" in (4.1) becomes "$=$"). For a given $\mathbf{f}$ (periodic in all d directions), we extend the solution $\mathbf{u}$ to $\mathbb{R}^d$ by periodicity and use a Fourier transform. We define $\mathbf{u}_\lambda$, $\mathbf{u}_\infty$, and $\mathbf{u}_0$ as the solutions to

$$A_\lambda \mathbf{u}_\lambda = \mathbf{f}, \quad A_\infty \mathbf{u}_\infty = \mathbf{f}, \quad \text{and } A\mathbf{u}_0 = \mathbf{f},$$

respectively. Here, A_∞ is the operator corresponding to the Stokes equation (with periodic boundary conditions) and $\mathbf{u}_\infty$ is the velocity component of its solution. To show that $M_\lambda = A_\lambda^{-1}$, we prove that

$$\mathbf{u}_\lambda = \frac{\lambda}{\lambda+1} \mathbf{u}_\infty + \frac{1}{\lambda+1} \mathbf{u}_0. \tag{4.7}$$

The proof of this relation is a straightforward computation using the Fourier transform and the following identities:

$$\widehat{\operatorname{div} \varepsilon(\mathbf{w})} = \frac{1}{2}|\xi|^2 (I + \Pi_\xi) \hat{\mathbf{w}}, \tag{4.8}$$

$$\widehat{\operatorname{div} C\varepsilon(\mathbf{w})} = \frac{1}{2}|\xi|^2 \left(I + (2\lambda+1)\Pi_\xi \right) \hat{\mathbf{w}}, \tag{4.9}$$

where $\Pi_\xi = |\xi|^{-2} \xi \xi^*$, $C(X) = \frac{1}{2}(X + X^*) + \lambda \operatorname{tr}(X) I$, and $X \in \mathbb{R}^{d \times d}$. Notice that $\Pi_\xi^2 = \Pi_\xi$ and, hence, (4.2) holds with Π_ξ instead of P. We then find that

$$\hat{\mathbf{u}}_\lambda = 2|\xi|^{-2} \left(I - \frac{2\lambda+1}{2(\lambda+1)} \Pi_\xi \right) \hat{\mathbf{f}}, \qquad \hat{\mathbf{u}}_0 = 2|\xi|^{-2} \left(I - \frac{1}{2}\Pi_\xi \right) \hat{\mathbf{f}}. \tag{4.10}$$

Furthermore, the Stokes' problem in the Fourier space is:

$$\begin{pmatrix} \frac{1}{2}|\xi|^2 (I + \Pi_\xi) & \xi \\ \xi^* & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{u}}_\infty \\ \hat{p} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{f}} \\ 0 \end{pmatrix}.$$

Solving this system shows that

$$\hat{\mathbf{u}}_\infty = 2|\xi|^{-2}(I - \Pi_\xi)\hat{\mathbf{f}}, \qquad \hat{p} = |\xi|^{-2}\xi^* \hat{\mathbf{f}}. \tag{4.11}$$

Finally, the relation (4.7) follows immediately from (4.10) and (4.11).

4.2. Discrete problems

Although we have defined the preconditioner M_λ for A_λ on the continuous level, a quick check shows that the analysis in Theorem 4.1 holds verbatim for the discretized problem as long as a Stokes stable finite-element pair, $V_h \times Q_h \subset V \times Q$, is available. In particular, assume $V_h \times Q_h$ satisfies the discrete inf-sup condition (cf. [27]),

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in V_h} \frac{(\operatorname{div} \mathbf{v}_h, q_h)}{\|\nabla \mathbf{v}_h\| \|q_h\|} \ge \beta_h > 0,$$

and let $A_\lambda^h : V_h \to V_h'$ and $\mathbf{u}_h \in V_h$ be given by

$$\langle A_\lambda^h \mathbf{u}_h, \mathbf{v}_h \rangle = a_\lambda(\mathbf{u}_h, \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle \quad \forall \mathbf{v} \in V_h. \tag{4.12}$$

Then, let $W_h = \{\mathbf{v}_h \in V_h : b(\mathbf{v}_h, q_h) = 0 \; \forall q_h \in Q_h\}$, $A_h = A_0^h$, and $P_h : V_h \to W_h$ be the $a(\cdot, \cdot)$ orthogonal projection.

An immediate, and important, observation is that computing the action of the preconditioner, M_λ^h, does not require a evaluating $P_h \mathbf{v}_h$ for some $\mathbf{v}_h \in V_h$ directly. This would be difficult, as a basis in the weakly divergence-free space, W_h, is not always available. Instead, the action of P_h is computed by solving the discrete Stokes' problem:

Find $P_h \mathbf{v}_h \in V_h$, $p_h \in Q_h$ such that

$$\begin{aligned}
a(P_h \mathbf{v}_h, \mathbf{w}_h) + b(\mathbf{w}_h, p_h) &= a(\mathbf{v}_h, \mathbf{w}_h), & \forall \mathbf{w}_h \in V_h, \\
b(P_h \mathbf{v}_h, q_h) &= 0, & \forall q_h \in Q_h.
\end{aligned} \tag{4.13}$$

We add that the analysis we have given here provides a theoretical justification for the preconditioning results reported in [28].

Remark 4.2. Under the assumption $\operatorname{div} V_h \subset Q_h$, (4.12) has uniform a priori error estimates for all $\lambda \in [0, \infty)$. Here we refer to [29–32] for stable Stokes' element pairs satisfying $\operatorname{div} V_h \subset Q_h$ and using discontinuous pressure spaces. We need to be careful when $\operatorname{div} V_h \not\subset Q_h$ and modify the bilinear form a_h^λ in such cases as follows:

$$\langle A_\lambda^h \mathbf{u}_h, \mathbf{v}_h \rangle = a(\mathbf{u}_h, \mathbf{v}_h) + \lambda b(\mathbf{v}_h, \Pi_h \operatorname{div} \mathbf{u}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle \quad \forall \mathbf{v} \in V_h, \tag{4.14}$$

where Π_h is the L^2-projection onto Q_h. This approach has been discussed in [1,11], where the role of Π_h is implicit but crucial to drawing the connection with the Stokes' equations, thus ensuring robust a priori error estimates. In general, the action of Π_h is computed by inverting a mass matrix, which could be costly, especially when the functions in Q_h are subject to inter-element continuity constraints. It is, however, easy to justify that we can use a spectrally equivalent diagonal matrix, such as the diagonal of the mass matrix, to implement the action of Π_h. This is the approach we have taken in the numerical tests for the Taylor-Hood [33] ($\mathcal{P}_2 \times \mathcal{P}_1$) element as presented in Section 5.

Based on Remark 4.2, a straightforward modification of the proof of Theorem 4.1 leads to the next theorem for the discretized problem (4.14).

Table 1
Number of iterations for $\mathcal{P}_2 \times \mathcal{P}_0$.

$h = 2^{-L}$	$\nu = 0.25$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.499$	$\nu = 0.4999$
$L = 2$	4	5	6	6	6
$L = 3$	3	4	6	7	7
$L = 4$	3	4	6	7	7
$L = 5$	3	4	6	7	7
$L = 6$	3	4	5	7	7

Theorem 4.3. *If $M_\lambda^h : V_h' \to V_h$ is defined by*

$$M_\lambda^h := \frac{\lambda}{1+\lambda} P_h A_h^{-1} + \frac{1}{\lambda+1} A_h^{-1}, \tag{4.15}$$

where A_λ^h is as defined in (4.14)*, then,*

$$\langle \mathbf{g}, M_\lambda^h \mathbf{g}\rangle \approx \langle \mathbf{g}, \left(A_\lambda^h\right)^{-1}\mathbf{g}\rangle \quad \forall \mathbf{g} \in V_h'.$$

Proof. First, from the proof of Theorem 4.1, we replace $\langle A_\lambda \mathbf{v}, \mathbf{v}\rangle = \langle A\mathbf{v}, \mathbf{v}\rangle + \lambda\|\operatorname{div}\mathbf{v}\|^2$ with its discretization:

$$\langle A_\lambda^h \mathbf{v}_h, \mathbf{v}_h\rangle = \langle A\mathbf{v}_h, \mathbf{v}_h\rangle + \lambda\|\Pi_h \operatorname{div}\mathbf{v}_h\|^2, \quad \forall \mathbf{v}_h \in V_h.$$

Then, the upper and lower bounds for $\|\Pi_h \operatorname{div}\mathbf{v}_h\|$ can be derived in analogous way to estimates in (4.5) and (4.6). This completes the proof. $\square$

Remark 4.4. Notice that when $\operatorname{div} V_h \not\subset Q_h$, an upper bound for $\|\operatorname{div}\mathbf{v}_h\|$ similar to (4.5) would not hold in general, thus validating the importance of introducing Π_h.

5. Numerical results

In this section, we provide numerical examples demonstrating the effectiveness and robustness of the preconditioner, M_λ^h, as defined in (4.15). The computational domain is $\Omega = (0,1)\times(0,1)$, and we seek to solve (2.1) for the exact solution $\mathbf{u}$ given by

$$\mathbf{u} = \left(\sin(\pi x)\cos(\pi y), -\cos(\pi x)\sin(\pi y)\right).$$

We compute the right-hand side, $\mathbf{f}$, accordingly and impose pure Dirichlet boundary conditions, i.e., $\Gamma = \Gamma_D$. Equations are discretized on a uniformly refined triangular mesh with mesh size $h = 2^{-L}$, where we use continuous and piecewise quadratic $\mathbb{R}^d$-valued polynomials in $\mathcal{P}_2$ to approximate $\mathbf{u}$. We test different finite-element spaces for the multiplier p.

The resulting linear system of equations is solved by the preconditioned conjugate gradient method, with M_λ^h as the preconditioner. We implement the actions of $P_h A_h^{-1}$ and A_h^{-1} using direct solvers. The stopping criterion is based on the relative residual with tolerance 10^{-6}. All numerical experiments, including the discretization and the preconditioned linear solvers, were implemented using the finite-element and solver library HAZmath [34].

For the first set of experiments, we employ the space of piecewise constants, $\mathcal{P}_0$, as the finite-element space for p. Thus, we implement the action of $P_h A_h^{-1}$ by solving Stokes' equations (3.7) using the $\mathcal{P}_2 \times \mathcal{P}_0$ finite-element pair, which is known to be inf-sup stable in 2D. We report the performance of the proposed preconditioner in Table 1 (for the number of iterations) and Table 2 (for the condition number of $M_\lambda^h A_\lambda^h$). These results show that the number of iterations and condition number remain stable as $\nu \to 0.5^-$, i.e., as $\lambda \to \infty$. This observation confirms our theoretical predictions.

For the second set of experiments, we utilize the nodal element space, $\mathcal{P}_1$, for p. In this case, as pointed out earlier in Remark 4.2, assembling $b(\mathbf{v}_h, \Pi_h\operatorname{div}\mathbf{u}_h)$ requires inverting $\mathcal{M}_h$, the $\mathcal{P}_1$ mass matrix. In the tests, however, we use the inverse of $\operatorname{diag}(\mathcal{M}_h)$ to approximate $\mathcal{M}_h^{-1}$. Then the action of $P_h A_h^{-1}$ is computed by solving (4.13) using the $\mathcal{P}_2 \times \mathcal{P}_1$ finite-element pair (Taylor-Hood). As Table 3 shows, the use of

Table 2
Condition number of $M_\lambda^h A_\lambda^h$ for $\mathcal{P}_2 \times \mathcal{P}_0$.

$h = 2^{-L}$	$\nu = 0.25$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.499$	$\nu = 0.4999$
$L = 2$	1.15	1.48	2.52	2.84	2.88
$L = 3$	1.14	1.44	2.47	2.98	3.03
$L = 4$	1.13	1.44	2.55	2.90	2.94
$L = 5$	1.13	1.44	2.51	2.86	2.89
$L = 6$	1.13	1.44	2.45	2.87	2.91

Table 3
Number of iterations for $\mathcal{P}_2 \times \mathcal{P}_1$ (without the projection Π_h).

$h = 2^{-L}$	$\nu = 0.25$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.499$	$\nu = 0.4999$
$L = 2$	6	11	25	34	46
$L = 3$	6	11	40	83	132
$L = 4$	6	11	40	126	206
$L = 5$	6	10	39	135	347
$L = 6$	5	9	37	135	410

Table 4
Number of iterations for $\mathcal{P}_2 \times \mathcal{P}_1$ (with the projection Π_h).

$h = 2^{-L}$	$\nu = 0.25$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.499$	$\nu = 0.4999$
$L = 2$	4	5	5	5	5
$L = 3$	4	6	11	12	12
$L = 4$	4	6	12	15	15
$L = 5$	4	6	12	15	15
$L = 6$	4	6	11	14	15

Table 5
Condition number $M_\lambda^h A_\lambda^h$ for $\mathcal{P}_2 \times \mathcal{P}_1$ (with the projection Π_h).

$h = 2^{-L}$	$\nu = 0.25$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.499$	$\nu = 0.4999$
$L = 2$	1.20	1.71	4.31	5.69	5.89
$L = 3$	1.20	1.71	4.38	5.81	6.02
$L = 4$	1.19	1.71	4.38	5.81	6.02
$L = 5$	1.18	1.71	4.38	5.81	6.02
$L = 6$	1.17	1.71	4.38	5.81	6.02

the projection, Π_h, i.e., using $b(\mathbf{v}_h, \mathbf{u}_h)$ directly, is essential for achieving a robust scheme. Whereas the number of iterations grows as $\nu \to 0.5^-$ without using the projection, we see in Table 4 (for the number of iterations) and Table 5 (for the condition number of $M_\lambda^h A_\lambda^h$) that with the projection Π_h, the performance of M_λ^h is greatly improved. Although the number of iterations and condition number is slightly higher than those obtained with the $\mathcal{P}_2 \times \mathcal{P}_0$ finite-element pair, they remain stable as $\nu \to 0.5^-$, i.e., as $\lambda \to \infty$. This indicates that the efficacy of the proposed preconditioner M_λ^h is not affected by the choice of finite-element space for p, as long as the corresponding finite-element pair is inf-sup stable.

6. Concluding remarks

Theorem 4.1 and the corresponding numerical tests in Section 5 confirm that a robust discretization and solvers for nearly incompressible elasticity must rely on robust solvers for Stokes' equations. This point is important as it confirms the relationship between the inf-sup condition for Stokes' equation and the second Korn's inequality for linear elasticity. Numerical results show that the proposed preconditioner, M_λ^h, remains stable as $\nu \to 0.5^-$, i.e., as $\lambda \to \infty$, regardless of the choice of finite-element space for p, as long as it forms a Stokes' inf-sup stable finite-element pair. We note that this includes higher-order approximations as well. While the preconditioner is robust for several families of finite elements, in our view, the best suited ones are elements recently developed in [35,31], as they provide spaces with projections that commute with the divergence and lead directly to discretizations for linear elasticity. Finally, we note that this work would be useful

in designing auxiliary space preconditioners for the elasticity equation when discretized using (any) stable finite-element space.

Data availability

Data will be made available on request.

Acknowledgements

Appendix A. Proof of Lemma 3.1

Lemma A.1. *Let $\mathfrak{R}$ be the space of rigid body motions*

$$\mathfrak{R} = \left\{ \mathbf{c} + \mathfrak{m}\mathbf{x} \mid \mathbf{c} \in \mathbb{R}^d, \quad \mathfrak{m} \in \mathfrak{so}(d) \right\},$$

where $\mathbf{x}$ is the position vector in $\mathbb{R}^d$ and $\mathfrak{so}(d)$ is the algebra of the real and anti-symmetric $d \times d$ matrices. Then it holds that

$$\|\nabla \mathbf{u}\| \lesssim \|\varepsilon(\mathbf{u})\|, \quad \forall \mathbf{u} \in [H_D^1(\Omega)]^d \cup \left([H^1(\Omega)]^d \cap \mathfrak{R}^{\perp_{L^2}} \right). \tag{A.1}$$

Proof. First, we only consider $\mathbf{u} \in V = [H_D^1(\Omega)]^d$ and remark that the proof for the case when $\mathbf{u} \in [H^1(\Omega)]^d \cap \mathfrak{R}^{\perp_{L^2}}$ is similar and simpler.

To start, assume that (A.1) is not true. Then, there exists a sequence $\{\mathbf{v}_n\} \subset V$ such that $\|\nabla \mathbf{v}_n\| = 1$ and $\|\varepsilon(\mathbf{v}_n)\| \leq \frac{1}{n}$. From the Poincaré inequality, we conclude that $\{\mathbf{v}_n\}$ is a bounded sequence in $L^2(\Omega)$. Next, since the embedding $V = [H_D^1(\Omega)]^d \hookrightarrow L^2(\Omega)$ is compact, we conclude that this bounded sequence has a subsequence convergent in $L^2(\Omega)$. We denote the subsequence again by $\{\mathbf{v}_n\}$. Applying (3.3) to $\mathbf{u} = (\mathbf{v}_n - \mathbf{v}_m)$ for sufficiently large n and m, we find that $\{\mathbf{v}_n\}$ is a Cauchy sequence in V and hence, converges to some element $\mathbf{v} \in V$. This gives $\|\nabla \mathbf{v}\| = 1$ and $\varepsilon(\mathbf{v}_n) \to 0$. Hence, $\varepsilon(\mathbf{v}) = 0$. This implies that $\mathbf{v}$ is a rigid body motion, namely, $\mathbf{v} = \mathfrak{m}\mathbf{x} + \mathbf{c} \in \mathfrak{R}$.

What remains is to show that if Γ_D has a nonzero $(d-1)$ dimensional measure, then $\mathbf{v} = 0$. This will lead to a contradiction with the assumption that (A.1) does not hold. Let us pick $\mathbf{x} \in \Gamma_D$ such that Γ_D is smooth in a neighborhood of $\mathbf{x}$. For the case of a polyhedral domain, which we consider here, take $\mathbf{x}$ in the interior of a planar face of Γ_D. For any $\mathbf{y}$ that is in this planar face, we have $\mathfrak{m}(\mathbf{x} - \mathbf{y}) = 0$. Since the face is of dimension $(d-1)$, it follows that $\mathfrak{m}$ has at least a $(d-1)$-dimensional kernel. However, $\mathfrak{m}$ is antisymmetric and real, and all its nonzero eigenvalues are pure imaginary and are complex conjugate to each other, that is, the nonzero eigenvalues come in pairs. Hence, we cannot have any nonzero eigenvalue of $\mathfrak{m}$. Thus, $\mathfrak{m} = 0$ and $\mathbf{v}$ is a constant vector vanishing on Γ_D. We then conclude that $\mathbf{v} = 0$ which contradicts $\|\nabla \mathbf{v}\| = 1$ and shows (A.1). $\square$

References

[1] D. Braess, Finite Elements: Theory, Fast Solvers, and Applications in Elasticity Theory, 3rd edition, Cambridge University Press, Cambridge, 2007, translated from German by Larry L. Schumaker.

[2] D. Boffi, F. Brezzi, M. Fortin, Mixed Finite Element Methods and Applications, Springer Series in Computational Mathematics, vol. 44, Springer, Heidelberg, 2013.

[3] J.H. Bramble, A proof of the inf-sup condition for the Stokes equations on Lipschitz domains, Math. Models Methods Appl. Sci. 13 (3) (2003) 361–371, https://doi.org/10.1142/S0218202503002544, dedicated to Jim Douglas, Jr. on the occasion of his 75th birthday.

[4] J. Douglas Jr., J.P. Wang, An absolutely stabilized finite element method for the Stokes problem, Math. Comput. 52 (186) (1989) 495–508, https://doi.org/10.2307/2008478.

[5] D. Boffi, C. Lovadina, Analysis of new augmented Lagrangian formulations for mixed finite element schemes, Numer. Math. 75 (4) (1997) 405–419, https://doi.org/10.1007/s002110050246.

[6] R. Glowinski, P. Le Tallec, Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics, SIAM Studies in Applied Mathematics, vol. 9, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989.

[7] H.C. Elman, D.J. Silvester, A.J. Wathen, Finite Elements and Fast Iterative Solvers: with Applications in Incompressible Fluid Dynamics, 2nd edition, Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, 2014.

[8] P.E. Farrell, L. Mitchell, F. Wechsung, An augmented Lagrangian preconditioner for the 3D stationary incompressible Navier–Stokes equations at high Reynolds number, SIAM J. Sci. Comput. 41 (5) (2019) A3073–A3096, https://doi.org/10.1137/18M1219370.

[9] J. Xia, P.E. Farrell, F. Wechsung, Augmented Lagrangian preconditioners for the Oseen–Frank model of nematic and cholesteric liquid crystals, BIT Numer. Math. 61 (2) (2021) 607–644, https://doi.org/10.1007/s10543-020-00838-9.

[10] F. Laakmann, P.E. Farrell, L. Mitchell, An augmented Lagrangian preconditioner for the magnetohydrodynamics equations at high Reynolds and coupling numbers, SIAM J. Sci. Comput. 44 (4) (2022) B1018–B1044, https://doi.org/10.1137/21M1416539.

[11] J. Schöberl, Multigrid methods for a parameter dependent problem in primal variables, Numer. Math. 84 (1) (1999) 97–119, https://doi.org/10.1007/s002110050465.

[12] C. Carstensen, A unifying theory of a posteriori finite element error control, Numer. Math. 100 (4) (2005) 617–637, https://doi.org/10.1007/s00211-004-0577-y.

[13] Y. Li, L. Zikatanov, A posteriori error estimates of finite element methods by preconditioning, Comput. Math. Appl. 91 (2021) 192–201, https://doi.org/10.1016/j.camwa.2020.08.001.

[14] D. Loghin, A.J. Wathen, Analysis of preconditioners for saddle-point problems, SIAM J. Sci. Comput. 25 (6) (2004) 2029–2049, https://doi.org/10.1137/S1064827502418203.

[15] K.-A. Mardal, R. Winther, Preconditioning discretizations of systems of partial differential equations, Numer. Linear Algebra Appl. 18 (1) (2011) 1–40, https://doi.org/10.1002/nla.716.

[16] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, Rev. Fr. Autom. Inform. Rec. Opér. Sér. Rouge 8 (R-2) (1974) 129–151.

[17] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, second English edition, revised and enlarged, Mathematics and Its Applications, vol. 2, Gordon and Breach, Science Publishers, New York-London-Paris, 1969.

[18] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer Series in Computational Mathematics, vol. 15, Springer-Verlag, New York, 1991.

[19] J. Nečas, Les méthodes directes en théorie des équations elliptiques, Masson et Cie, Éditeurs, Paris, 1967, Academia, Éditeurs, Prague.

[20] A. Korn, Solution general du probleme d'equilibre dans la theorie de l'elasticite, Ann. Fac. Sci. Toulouse 10 (1908) 705–724.

[21] A. Korn, Ueber einige Ungleichungen, welche in der Theorie der elastischen und elektrischen Schwingungen eine Rolle spielen, Bulle. Int. Acad. Sci. Cracov. 9 (1909) 705–724.

[22] V.A. Kondratiev, O.A. Oleinik, On Korn's inequalities, C. R. Acad. Sci., Sér. 1 Math. 308 (16) (1989) 483–487, https://doi.org/10.1070/RM1989v044n06ABEH002297.

[23] V.A. Kondratiev, O.A. Oleinik, Dependence of the constants in the Korn inequality on parameters that characterize the geometry of the domain, Usp. Mat. Nauk 44 (6(270)) (1989) 157–158, https://doi.org/10.1070/RM1989v044n06ABEH002297.

[24] G. Duvaut, J.-L. Lions, Inequalities in Mechanics and Physics, translated from the French by C.W. John, Grundlehren der Mathematischen Wissenschaften, vol. 219, Springer-Verlag, Berlin-New York, 1976.

[25] J.A. Nitsche, On Korn's second inequality, RAIRO. Anal. Numér. 15 (3) (1981) 237–248.

[26] D.N. Arnold, Linear plate theory: modeling, analysis, and computation, in: Lecture Notes for Special Topic Course Taught by D.N. Arnold at Penn State, Fall 1997, 1997.

[27] V. Girault, P.-A. Raviart, Finite element methods for Navier-Stokes equations: theory and algorithms, in: Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, Berlin, 1986.

[28] E. Karer, J.K. Kraus, L.T. Zikatanov, A subspace correction method for nearly singular linear elasticity problems, in: Domain Decomposition Methods in Science and Engineering XX, in: Lect. Notes Comput. Sci. Eng., vol. 91, Springer, Heidelberg, 2013, pp. 159–166.

[29] J. Guzmán, M. Neilan, Conforming and divergence-free Stokes elements on general triangular meshes, Math. Comput. 83 (285) (2014) 15–36, https://doi.org/10.1090/S0025-5718-2013-02753-6.

[30] V. John, A. Linke, C. Merdon, M. Neilan, L.G. Rebholz, On the divergence constraint in mixed finite element methods for incompressible flows, SIAM Rev. 59 (3) (2017) 492–544, https://doi.org/10.1137/15M1047696.

[31] S.H. Christiansen, K. Hu, Generalized finite element systems for smooth differential forms and Stokes' problem, Numer. Math. 140 (2) (2018) 327–371, https://doi.org/10.1007/s00211-018-0970-6.

[32] G. Fu, J. Guzmán, M. Neilan, Exact smooth piecewise polynomial sequences on Alfeld splits, Math. Comput. 89 (323) (2020) 1059–1091, https://doi.org/10.1090/mcom/3520.

[33] C. Taylor, P. Hood, A numerical solution of the Navier-Stokes equations using the finite element technique, Int. J. Comput. Fluids 1 (1) (1973) 73–100, https://doi.org/10.1016/0045-7930(73)90027-3.

[34] J.H. Adler, X. Hu, L.T. Zikatanov, HAZmath: a simple finite element, graph, and solver library (2009-present), URL https://github.com/HAZmathTeam/hazmath.

[35] S.H. Christiansen, K. Hu, A finite element for Stokes with a commuting diagram, Mathematical Analysis in Fluid and Gas Dynamics, RIMS Kōkyūroku(2107), 2019, pp. 172–183, https://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/pdf/2107-11.pdf.