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GRADED MESH APPROXIMATION IN WEIGHTED SOBOLEV SPACES AND ELLIPTIC EQUATIONS IN 2D

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Abstract. We study the approximation properties of some general finite-element spaces constructed using improved graded meshes. In our results, either the approximating function or the function to be approximated (or both) are in a weighted Sobolev space. We consider also the L^p-version of these spaces. The finite-element spaces that we define are obtained from *conformally invariant* families of finite elements (no affine invariance is used), stressing the use of elements that lead to higher regularity finite-element spaces. We prove that for a suitable grading of the meshes, one obtains the usual optimal approximation results. We provide a construction of these spaces that does not lead to long, "skinny" triangles. Our results are then used to obtain L^2-error estimates and h^m-quasi-optimal rates of convergence for the FEM approximation of solutions of strongly elliptic interface/boundary value problems.

Introduction

Consider the typical problem of approximating the solutions of a mixed boundary value diffusion problem with zero Dirichlet and Neumann boundary conditions,

$$\begin{cases} -\operatorname{div}(A\nabla u) = f & \text{in } \Omega, \\ \nu \cdot A \cdot \nabla u = 0 & \text{on } \partial_N \Omega, \\ u = 0 & \text{on } \partial_D \Omega. \end{cases} \tag{0.1}$$

Here, ν is the outward normal vector to the boundary, Ω is a polygonal domain in two space dimensions (2D) with straight faces and $f \in H^{m-1}(\Omega)$. We allow piecewise smooth coefficients, so that transmission (or interface) problems can also be considered. This problem arises in many practical applications. Typically one is interested in approximating the solution, u, with some simpler functions or at least in approximately computing some quantities of interest associated to u. One of the most commonly used methods to approximate u is the Finite-Element Method (FEM). See for instance [8, 14, 18, 23, 42, 44] for introductions to this method. The results here extend almost without change to systems such as the elasticity system and to nonhomogeneous boundary conditions. We can also include lower order terms, as long as the basic regularity and well posedness properties of equation (0.1) are preserved. We also extend certain results of this paper to general elliptic problems of order 2μ ($\mu \geq 1$).

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The standard applications of FEM to approximate the solution, u, of equation (0.1) requires it to possess good regularity properties. However, it is known [7,28, 29,32] that the solution, u, of this equation on non-smooth domains is typically *not* in $H^{m+1}(\Omega)$, but rather has the limited regularity, $u \in H^{s-\epsilon}(\Omega)$, where $s = s_\Omega$ is a constant associated to Ω and where $\epsilon > 0$ is arbitrary. This is not an artifact of the method, but it does lead to decreased rates of convergence in the FEM approximations. Due to its practical importance, many works have been devoted to correcting this deficiency; see for example [1,4,9–14,25,27,30,34,37].

A common result of the above research is that the solution, u, of equation (0.1) has better regularity properties in a weighted Sobolev space [32,33,37]. More precisely, let $r_\Omega(x)$ denote the distance from x to the set of singular points of Ω [37] and define the L^p-based weighted Sobolev space of order m and weight a by

$$\mathcal{W}_a^{m,p}(\Omega) = \{v : \Omega \to \mathbb{C},\ r_\Omega^{|\alpha|-a}\partial^\alpha v \in L^p(\Omega),\ |\alpha| \le m\}. \tag{0.2}$$

For the most part, we consider $p = 2$ and denote $\mathcal{W}_a^{m,2}(\Omega) = \mathcal{K}_a^m(\Omega)$, which are Hilbert spaces. However, in general, L^p-based weighted Sobolev spaces make the results useful for some nonlinear problems and some of the results are extended to these cases. Then, if $f \in H^{m-1}(\Omega)$, we have that $u \in \mathcal{W}_{a+1}^{m+1,2}(\Omega) = \mathcal{K}_{a+1}^{m+1}(\Omega)$, for $a > 0$ small enough. (More precisely, in the case of the Poisson problem with only Dirichlet boundary conditions, $a < \pi/\alpha_{\max}$, where $\alpha_{\max} \in (0, 2\pi]$ is the maximum angle of Ω). In two dimensions, this is enough to recover quasi-optimal rates of convergence for the finite-element approximations of u [1,9,12,37]. The proof of this result depends essentially on an approximation property of the solution, u, in weighted Sobolev spaces using graded meshes. More precisely, it was shown (see for example [3,9,12,37]) that there exists a sequence of nested meshes, $\mathcal{T}_n$, with the following property: Let S_n be the sequence of finite-element spaces of continuous functions on Ω that are restricted on each triangle of $\mathcal{T}_n$ to a polynomial of order m. Then, $\dim(S_n) \to \infty$ and there exists a sequence of interpolation operators I_n and a constant $C > 0$ with the property $\|u - I_n(u)\|_{H^1(\Omega)} \le C \dim(S_n)^{-m/2}\|u\|_{\mathcal{K}_{a+1}^{m+1}(\Omega)}$, for $m \ge 1$. Since there exists a constant C such that $\|u\|_{\mathcal{K}_{a+1}^{m+1}(\Omega)} \le C\|f\|_{H^{m-1}(\Omega)}$, the above approximation becomes

$$\|u - I_n(u)\|_{H^1(\Omega)} \le C_a \dim(S_n)^{-m/2}\|f\|_{H^{m-1}(\Omega)}, \tag{0.3}$$

for some constant $C_a > 0$. This is the same result that one would obtain in the classical case of quasi-uniform meshes, if the solution u were in $H^{m+1}(\Omega)$. See also [38], where several numerical tests and comparisons with other methods were provided.

In this paper, we extend the approximation result of equation (0.3) above in several ways and offer different variants of the construction of the sequence of graded meshes $\mathcal{T}_n$. For instance, we offer a construction that yields a minimum angle condition *independent* of m. This is relevant for the hp-version of the FEM as long skinny triangles are avoided. We also include interfaces, in the sense that we partition the domain, Ω, into several straight polygonal subdomains, Ω_j, and only assume $u \in \mathcal{K}_{a+1}^{m+1}(\Omega_j)$ for all j, but $u \in \mathcal{K}_{a+1}^1(\Omega)$ for the entire domain.

Another new feature presented in this paper is that we consider finite-element spaces with higher regularity, forcing one to consider finite-element spaces generated by *conformally invariant families* (this definition is slightly more general than the one in Section 2.3 of [23], for instance). Therefore, the constructions are not restricted to Lagrange finite elements. This may be useful for higher order problems, so error estimates are also provided in higher Sobolev norms. See [17, 20, 31, 40], for example. In addition, estimates in L^p-based Sobolev norms are provided, which may be useful for nonlinear problems. Finally, in the application to the Finite Element Method for the second order problem, (0.1), we also provide L^2-error estimates using the Aubin-Nitsche trick.

To formulate the general form of problem (0.1) more precisely, assume that the straight polygonal domain, Ω, is decomposed as $\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k$, where Ω_k are disjoint straight polygonal domains. The set $\Gamma := \partial\Omega \smallsetminus \bigcup_{k=1}^{K} \partial\Omega_k$, that is, the part of the boundary of some Ω_k that is not contained in the boundary of Ω, is called the *interface*, as usual. Then, assume that the coefficients $A = [a_{ij}]$ have only jump discontinuities across the interface Γ. That is, the restriction to any of the domains, Ω_k, of any of the coefficients, a_{ij}, extends to a smooth function on $\overline{\Omega}_k$. Also, equation (0.1) is formulated in a weak sense, which implies the usual matching and jump conditions at the interface. (See Section 3 for details.) We also assume that $-\operatorname{div}(A\nabla)$ is uniformly strongly elliptic, in the usual sense; see equation (3.2).

The paper is organized as follows. In Section 1, we describe the family of finite elements that we consider and we also state the outline of the problem. In particular, we consider more general finite elements that are not necessarily affine equivalent in order to obtain higher regularity approximation spaces. In Section 2, we state and prove the approximation results. We state these for general elliptic problems of order 2μ, $\mu \geq 1$, and in general L^p-based Sobolev spaces when possible. For $\mu = 1$ and $p = 2$, we extend our approximation results to some augmented Sobolev spaces that arise in our treatment of transmission and mixed boundary value problems. In addition, we describe the graded meshes and how they produce optimal results for these types of problems with singularities. Section 3 gives examples on how the above results can be applied to finite-element discretizations for certain problems, such as transmission/interface and mixed boundary value problems. This includes an L^2-error estimate as well as a description on how to obtain classical "textbook" h^m-error estimates. Finally, we make some concluding remarks toward the end of the paper.

1. Conformal families of finite elements

Consider a bounded polygonal domain, $\Omega \subset \mathbb{R}^2$, with *straight* edges. Also assume that a disjoint decomposition of the boundary into "Dirichlet" and "Neumann" parts, $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$, is given with $\partial_D\Omega$ a closed subset, and both $\partial_D\Omega$ and $\partial_N\Omega$ sets with finitely many components.

This section explains the type of finite elements considered in this paper. The reader may skip this section at the first reading and go to the next section, assuming for instance that order m Lagrange elements are used, in which case the construction is simpler.

1.1. **A typical finite element.**

Consider (essentially) the framework of Ciarlet [23, 24]. The main difference in our approach is that we emphasize more the interpolant rather than the degrees of freedom. Also, the families will not be, in general, affine equivalent, so we do not consider reference finite elements.

Consider an arbitrary triangle T, fixed throughout this subsection. Denote by $\mathcal{P}_j$, the space of polynomials $P : \mathbb{R}^2 \to \mathbb{R}$ of degree (at most) j, and fix a dimension-N space $\mathcal{P}_T$,

$$\mathcal{P}_m \subset \mathcal{P}_T \subset \mathcal{P}_{m'}. \tag{1.1}$$

In addition to the integer parameters m and m' introduced above in equation (1.1), we also fix two integers $s' \leq s$. The integer s determines the smoothness of the functions that can be interpolated (by the higher order of the derivatives appearing among the degrees of freedom) and is called the *degree of the interpolant*. The integer s' is defined such that the resulting finite-element (FE) space has smoothness $H^{s'+1}$. The integer m is called the *polynomial degree of approximation* and the integer m' is called the *maximum polynomial degree of the FE spaces*.

Then, consider linear functionals, ℓ_i, on $\mathcal{C}^s(\mathbb{R}^2)$, $1 \leq i \leq N = \dim(\mathcal{P}_T)$, called *degrees of freedom*, whose restrictions to $\mathcal{P}_T$ form a basis of $\mathcal{P}_T^*$, where $\mathcal{P}_T^*$ denotes the space of linear functionals on $\mathcal{P}_T$. Assume that the degrees of freedom are of one of the following three types, with q_1 and q_2 denoting polynomials:

$$\begin{gathered}
\ell_i : \mathcal{P}_T \to \mathbb{R}, \quad \ell_i(q_1) = \partial_x^j \partial_y^k q_1(z), \quad \text{where } z \in T \text{ and } \ j+k \leq s\,, \\
\ell_i(q_1) = \int_\sigma q_2(x,y) \partial_\nu^k q_1(x,y) d\sigma\,, \quad \text{where } \sigma = \ \text{an edge of } T \ \text{ and } k \leq s, \text{ or} \\
\ell_i(q_1) = \int_T q_2(x,y) q_1(x,y) dxdy.
\end{gathered} \tag{1.2}$$

If ℓ_i is of the first type (type I), then we say that its support is z. If ℓ_i is of the second type (type II), then we say that its support is σ. Finally, if ℓ_i is of the third type (type III), we say that its support is T. Thus, the support of ℓ_i is the support of the corresponding distribution. If ℓ_i is of type II, then ν is a unit normal vector to σ and $d\sigma$ is the arc length measure on σ. The motivation for considering degrees of freedom that are not type I is provided, for instance, by [5,6].

Denote by $\Sigma_T := \{\ell_i\}$ the set of the given linear functionals, which are assumed to be a linearly-independent set. Since Σ_T is a basis of $\mathcal{P}_T^*$, there exists a dual basis $\{q_i\} \subset \mathcal{P}_T$ such that

$$\ell_i(q_j) = \delta_{ij}\,,$$

and, then, we define the interpolation operator as usual,

$$I_T = I_{T,\mathcal{P}_T,\Sigma} : \mathcal{C}^s(\overline{T}) \to \mathcal{P}_T, \quad I_T(f) = \sum_{i=1}^N \ell_i(f) q_i. \tag{1.3}$$

(Recall that δ_{ij} denotes the Kronecker symbol: $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$.) The fact that $\{q_i\}$ is the dual basis to $\{\ell_i\}$ ensures that $I_T^2 = I_T$, and, hence, I_T is a projection onto $\mathcal{P}_T$, that is $I_T(q) = q$, for all $q \in \mathcal{P}_T$. Then, $\Xi_T := (T, \mathcal{P}_T, I_T)$ is a *finite element* on T. The *support of* Ξ_T is the set consisting of the supports of the degrees of freedom $\ell_i \in \Sigma_T$. Note, however, that the set Σ_T is not determined by the triple $\Xi_T := (T, \mathcal{P}_T, I_T)$, in the sense that the same finite element can be obtained from a different choice of degrees of freedom. It is in this sense that we choose to make I_T a more important ingredient of the definition than the set Σ_T. The reason for this choice is the better invariance properties of the interpolation operator, I_T, than those of the set Σ_T; see below. This is the main difference to [23]. Note that the support of Ξ_T is independent on the choice of the degrees of freedom.

For any open subset $U \subset \mathbb{R}^d$, denote

$$W^{b,p}(U) = \{v : U \to \mathbb{C},\, \partial^\alpha v \in L^p(\Omega),\, |\alpha| \le b\}\,, \quad b \in \mathbb{Z}_+\,, \tag{1.4}$$

as the nonweighted analogues of the spaces $\mathcal{W}^{b,p}(\Omega)$ introduced in equation (0.2). Then, $W^{b,p}(\mathbb{R}^d) \subset \mathcal{C}^s(U)$ for $b > s + d/p$, by the Sobolev embedding theorem. In particular, we obtain for $b > s + d/p$ that I_T introduced earlier also defines a linear map,

$$I_T = I_{T,\mathcal{P}_T,\Sigma} : W^{b,p}(\mathbb{R}^2) \to \mathcal{P}_T\,. \tag{1.5}$$

Finally, if T' is any other triangle and L is an invertible affine map (that is a linear map plus a translation) such that $L(T) = T'$, then L can transport $\mathcal{P}_T$, I_T, and the support of $\Xi_T = (T, \mathcal{P}_T, I_T)$ to T' as usual. Explicitly, let

$$\mathcal{P}_{T'} := \{q \circ L^{-1}, q \in \mathcal{P}_T\} \quad \text{and} \quad I_{T'}(f) = I_T(f \circ L) \circ L^{-1}. \tag{1.6}$$

Then, denote by $L(I_T) = I_{T'}$, $L(\mathcal{P}_T) = \mathcal{P}_{T'}$, and

$$L(\Xi_T) := (L(T), L(\mathcal{P}_T), L(I_T)) = (T', \mathcal{P}_{T'}, I_{T'}) = \Xi_{T'}, \tag{1.7}$$

the corresponding finite element on T'.

1.2. **Assumptions on the families of finite elements.** Assume that to each triangle, T, in the plane there is associated a finite element $\Xi_T = (T, \mathcal{P}_T, I_T)$ and that this family depends continuously on T (in an obvious sense; see Assumption 1). The following three assumptions are made for this family, $\mathcal{F} = \{\Xi_T\}$, of finite elements. First, in order to obtain interpolation estimates in the usual way, we assume that the family is *continuous and conformally invariant,* two conditions that are defined shortly.

In order to introduce the continuity condition, we first identify the set of triangles in the plane as a subset of the set of triples $\{(A, B, C) : A, B, C \in \mathbb{R}^2\}/S_3$, where the quotient by the group, S_3, means that we identify two triples if they differ by a permutation. Here, S_3 is the set of permutations of three elements. The set of triangles are, therefore, a subset of $\mathbb{R}^6/S_3$. Let T_0 be a fixed (reference) triangle. Then, for any other triangle, T, there exists *exactly* six affine maps $L_T : T \to T_0$, such that $L_T(T) = T_0$ (they all differ by a permutation of the vertices of T_0). Next, fix a triangle, T, and a small neighborhood, U, around it. Then, if the neighborhood, U, is small enough, we can choose a consistent (continuous) ordering of the vertices of the triangles, T' in U. For example, if the triangle has (distinct) vertices, A, B, and C, we consider the neighborhood defined by all triples (A', B', C'), where $|AA'|, |BB'|, |CC'| < \epsilon$, with $\epsilon < \min\{|AB|, |BC|, |CA|\}/2$. By choosing an ordering on the vertices of T_0 as well, we obtain a unique choice of an affine map, $L_{T'}(T') = T_0$, for $T' \in U$, by requiring that $L_{T'}$ preserve the order of the vertices. Then, the family, $L_{T'}$, depends continuously on T'. If $T = T_0$, we arrange the vertices, such that $L_T = I$ and then all the other affine maps, $L_{T'}$, are close to the identity transformation, I. Recall that $L_{T'}(I_{T'})$ is defined by $L_{T'}(I_{T'})(f) = I_{T'}(f \circ L_{T'}) \circ L_{T'}^{-1}$, as in equation (1.6).

Definition 1.1. We say that the family, $\mathcal{F}$, has the *continuity property*, if for any fixed T and a suitable small neighborhood U of T that provides a continuous choice of affine maps, $L_{T'}(T') = T_0$ for $T' \in U$, we have that the resulting family of interpolants (linear maps), $L_{T'}(I_{T'}) \in \mathcal{L}(\mathcal{C}^s(T_0), \mathcal{P}_{m'})$, depends continuously on $T' \in U$.

Since all choices of the family, $L_{T'}$, differ by a permutation of the vertices of T_0, we have that the above definition makes sense (that is, it is independent of the choice of the family, $L_{T'}$).

In addition, we need the natural definition of a *conformal invariance* for the family, $\mathcal{F}$. As usual, we say that $\mathcal{F}$ is conformally invariant if for any conformal linear map, L (the composition of dilations and isometries), such that $L(T) = T'$, then $L(\Xi_T) = \Xi_{T'}$.

Assumption 1: Continuity and conformal invariance. *We assume that the family, $\mathcal{F} = \{\Xi_T\}$, $\Xi_T = (T, \mathcal{P}_T, I_T)$, has the continuity property and is conformally invariant.*

Notice that the continuity condition implies the continuity of the supports of the elements, Ξ_T. In particular, if one of the finite elements has an edge in its support, then all the edges of all triangles are in the supports of the corresponding Ξ_T. Next, we want the resulting finite-element spaces to have smoothness $\mathcal{C}^{s'}$.

Assumption 2: Matching of derivatives. *Let σ be an edge and choose any triangle T with σ as an edge. Also, denote by $\Sigma_{T,\sigma}$ the set of degrees of freedom $\ell_i \in \Sigma_T$ with support contained in σ. We assume that if $q \in \mathcal{P}_T$ is such that $\ell_i(q) = 0$ for all $\ell_i \in \Sigma_{T,\sigma}$, then $\partial_\nu^k q = 0$ on σ for all $k \leq s'$ (here, as before, ν is the normal derivative to σ).*

The third assumption has to do with constructing degrees of freedom of the associated finite-element space. Let T be a triangle with associated finite element Ξ_T. Fix a set of degrees of freedom, Σ_T, defining Ξ_T. We shall regard Σ_T as a subset of the space, $\mathcal{C}^s(\mathbb{R}^2)'$, of continuous linear maps $\mathcal{C}^s(\mathbb{R}^2) \to \mathbb{R}$. In order to formulate the assumption on the meshes, we need to introduce the following notation:

(1) Let z be a vertex of T, then denote V_z as the linear span of the degrees of freedom supported on z.
(2) Let σ be an edge of T, then denote V_σ as the linear span of the set of degrees of freedom of Σ_T supported on σ.
(3) Let σ be an edge of T and $z \in \sigma$ be a point that is not a vertex of σ, then denote $V_{\sigma,z}$ as the linear span of the set of degrees of freedom of Σ_T supported on z.
(4) Finally, denote V_T as the set of degrees of freedom of Σ_T supported on T.

Then, the third assumption on the family $\mathcal{F}$ of finite elements is as follows.

Assumption 3: Independence of degrees of freedom. *The sets V_z, V_σ, and $V_{\sigma,z}$, introduced above, do not depend on T (they depend only on the indicated subscripts).*

1.3. **Examples of families of finite elements.** We now construct examples of families, $\mathcal{F} = \{\Xi_T\}$, $\Xi_T = (T, \mathcal{P}_T, I_T)$, of finite elements satisfying the three assumptions of the previous subsection, where T ranges through the set of all triangles.

The most important example is also the simplest: *Lagrange triangles of type* (m) [18, 23]. Recall that in this case, $m = m'$ and $s' = s = 0$, so $\mathcal{P}_T = \mathcal{P}_m$ for any triangle T. The degrees of freedom are given by the evaluations at the points $z \in T$ that, in barycentric coordinates on T, are of the form $[\lambda_0/m, \lambda_1/m, \lambda_2/m]$, where λ_1, λ_2, and λ_3 are integers. In fact, these families are even affine invariant (not just conformally invariant) [18, 23], and affine invariant families have the continuity property of Definition 1.1.

Higher regularity finite-element spaces are needed for fourth order problems, such as the bi-Laplacian [39, 43] and in certain formulations of second order elliptic problems using the least-squares finite-element method [13, 15, 26, 34]. Two examples that yield $\mathcal{C}^1$-finite elements are the Argyris and the Bell triangle [23]. The *Argyris triangle* provides an example with $m = m' = 5$, $s' = 1$, and $s = 2$. The degrees of freedom are all the partial derivatives of order $\leq s = 2$ at the vertices and the normal derivative at the midpoints of the edges (so $N = 21$). The *Bell triangle* is similar, but $m = 4$ (m', s, and s' are the same). Both the Argyris and Bell triangles require $\mathcal{C}^2$-regularity of the function to be approximated (in the domain of the interpolant), while yielding only $\mathcal{C}^1$-regularity for the resulting finite-element space. Since the degrees of freedom in the Argyris and Bell triangles depend only on the geometry of the triangle, and not on the particular coordinate system, these families of finite elements satisfy Assumptions 1 and 2. Thus, the results in this paper apply to these families.

1.4. **Finite-element space and interpolation.** With the notation and the assumptions of Subsection 1.2, let $\mathcal{F}$ denote the given family of finite elements Ξ_T. We now extend the constructions of that subsection to a mesh, $\mathcal{T} = \{T_j\}$, on Ω yielding a finite-element space, $S(\mathcal{T}, \mathcal{F}) = S(\mathcal{T})$, and an interpolation operator, $I_{\mathcal{T},\mathcal{F}} = I_{\mathcal{T}}$, as follows. Recall, that a *mesh* $\mathcal{T}$ on Ω is a set of disjoint (open) triangles, $T_j \subset \Omega$, satisfying $\bigcup \overline{T}_j = \overline{\Omega}$. We also assume that $\partial_D\Omega$, the Dirichlet part of the boundary, is a union of edges of triangles, $T \in \mathcal{T}$. Hence, the Neumann part of the boundary, $\partial_N\Omega$, has the same property. Eventually (beginning with Section 2), we require the triangles of $\mathcal{T}$ to be *aligned to the interface*. For this discussion, though, this is not necessary. We also assume that the meshes are *conforming*, meaning, as usual, that if the closures $\overline{T_i}$ and $\overline{T_j}$ of two triangles T_i and T_j of this mesh intersect, then their intersection $\overline{T_i} \cap \overline{T_j}$ is either a vertex or an edge of these triangles.

Consider for each triangle, T, the space $\mathcal{P}_T$ and the support of the finite element Ξ_T. The union of the supports of all the finite elements, Ξ_T, is the support of $\mathcal{T}$. Recall the notation introduced in Subsection 1.2 and consider the following. For each vertex z of $\mathcal{T}$, consider a basis Σ_z of the set V_z. For each edge σ of $\mathcal{T}$, consider a basis Σ_σ of V_σ. Finally, for each edge σ of $\mathcal{T}$ and $z \in \sigma$ that is the support of some degree of freedom, consider a basis $\Sigma_{\sigma,z}$ of $V_{\sigma,z}$. Notice that the sets Σ_z, Σ_σ, and $\Sigma_{\sigma,z}$ are defined using a triangle T of the mesh. However, by Assumption 3 of Subsection 1.2, these sets do not depend on the choice of T. Then, the set $\Sigma_{\mathcal{T},\mathcal{F}}$ of degrees of freedom associated to $\mathcal{T}$ and $\mathcal{F}$ is defined as the union of the sets Σ_z, Σ_σ, $\Sigma_{\sigma,z}$ considered above, and of the basis of all the spaces V_T, where T ranges through all the triangles of $\mathcal{T}$.

We now define the finite-element space, $S(\mathcal{T}) = S(\mathcal{T}, \mathcal{F})$, associated to $\mathcal{F} := \{\Xi_T = (T, \mathcal{P}_T, \Sigma_T)\}$ and $\mathcal{T}$ as follows. Consider the set of families, (f_T), $f_T \in \mathcal{P}_T$, for all $T \in \mathcal{T}$. Such a family, (f_T), is called *matching* if $\ell_i(f_T) = \ell_i(f_{T'})$ and if T and T' are two adjacent triangles of $\mathcal{T}$, such that $\ell_i \in \Sigma_{\mathcal{T},\mathcal{F}}$ is a degree of freedom that is common to both T and T' (that is, it has support contained in the intersection of $\overline{T}$ with $\overline{T'}$). Then,

$$S(\mathcal{T}, \mathcal{F}) := \{(f_T),\ f_T \in \mathcal{P}_T \text{ is a matching family}\}\,. \tag{1.8}$$

Since this paper also deals with higher order equations, it is convenient to impose the boundary conditions in the approximation spaces later on. The plan is to obtain the approximation results for the interpolant without imposing boundary conditions and, then, to show that the resulting interpolant does satisfy the boundary conditions.

Let $u \in \mathcal{C}^s(\Omega)$. Then, for each triangle T in the mesh $\mathcal{T}$, the interpolant $u_T := I_T(u) \in \mathcal{P}_T$ is defined, where I_T is associated to T and the family $\mathcal{F} := \{\Xi_T\}$, $\Xi_T = (T, \mathcal{P}_T, I_T)$, as before. The assumptions from Subsection 1.2 on the finite element shows that the collection (u_T) is a matching family, so it is in $S(\mathcal{F})$. We then define

$$I_{\mathcal{T},\mathcal{F}}(u) = (I_T(u)) \in S(\mathcal{T}, \mathcal{F}). \tag{1.9}$$

2. Discretization error estimates and κ-refinements

The purpose of this section is to describe a sequence of (graded) triangular meshes, $\mathcal{T}_n$, in the domain, Ω, that provide quasi-optimal approximations of functions in suitable weighted Sobolev spaces. In particular, we extend the approximation results of [12, 37] from Lagrange elements to the more general elements described above. We also consider approximations in other norms. Let $m \le m'$ and $s' \le s$ be as in the previous section. That is, the integer m is the polynomial degree of approximation, the integer m' is the maximum polynomial degree of the FE spaces, the integer s is the degree of the interpolant, and the integer $s' + 1$ is the smoothness of the FE space. Also, we consider a continuous family, $\mathcal{F}$, of *conformally invariant* finite elements, as described above in Subsection 1.2.

Next, consider the interfaces. Recall that the domain, Ω, is a polygonal domain with straight edges (called a *straight polygon*). For simplicity, we do not allow for cracks or vertices that touch the boundary. The case of cracks would be very similar to that of the interface, but would allow functions with jump discontinuities along the crack. We leave it to the reader to make the necessary changes to deal with cracks. On the other hand, when considering mixed boundary conditions, it is well known that singularities appear at the points where the boundary conditions change (from Dirichlet to Neumann). These singularities are very similar in structure to the singularities that appear at geometric vertices. Thus, we define the *set $\mathcal{V}$ of singular points* of Ω as the set where singularities of the solutions of elliptic partial differential equations may arise. These consist, in our case, of all the geometric vertices of Ω, all points where the type of boundary conditions change, all points where the interface touches the boundary, and all the nonsmooth points of the interface. (If cracks were allowed, then the tips and ends of the cracks would be included as well as the nonsmooth points of the cracks).

Assume that $\overline{\Omega} = \bigcup_{j=1}^{K} \overline{\Omega_j}$, with Ω_j also straight polygons. Assume the domains, Ω_j, are open and disjoint. For the numerical solution of the problem (0.1), we are looking to approximate a function $u \in \mathcal{K}^{\mu}_{a+b+\mu}(\Omega)$, such that the restriction of u to any of the subdomains, Ω_j, satisfies $u \in \mathcal{K}^{m+\mu}_{a+b+\mu}(\Omega_j)$ for all j. However, here, we think of u as the solution of an elliptic problem of order 2μ, generalizing problem (0.1), and of $a > 0$ as a constant that depends on that problem. To formulate the approximation results, we do not need u to be a solution of an elliptic problem, but only to belong to a suitable weighted Sobolev space. Moreover, we consider L^p-based Sobolev spaces $\mathcal{W}^{m+\mu,p}_{a+b+\mu}(\Omega)$, in view of possible applications to nonlinear problems, as suggested by one of the referees. Also, $b \in \mathbb{R}$ is a parameter that is considered in view of further applications to the least squares finite-element method [34]. It is, therefore, convenient to assume that the initial refinement of Ω is such that the interface is *resolved exactly* by the meshes, that is, the interface is a union of the edges of the meshes that are considered. It turns out that it is enough to assume that the initial mesh, $\mathcal{T}_0$, resolves the interface exactly.

For the remaining part of this paper, the assumptions that the boundary of Ω and the interface Γ are piecewise linear will be crucial, in order to avoid approximating the boundary and the interface with finite elements, which is an important, but completely different problem. For simplicity of the theoretical analysis, we again assume that there are no cracks. We note, however, that the mesh refinement in the case of cracks is completely similar, as long as one resolves the crack exactly and allows for discontinuous approximation of functions along it. (From an implementation point of view, this amounts to doubling the points on the crack, except at the tips.)

2.1. **Approximation away from the singular points.** We start by discussing the simpler approximation of the solution, u, *away* from the singular points. Recall that all estimates in the spaces $\mathcal{K}_a^m(\Omega)$ localize to subsets of Ω.

Let $\mathbb{P}$ be any polygonal domain. In the applications considered here, $\mathbb{P}$ is a subset of Ω. Let $\mathcal{T}$ be a mesh of $\mathbb{P}$ and let $S(\mathcal{T}, \mathcal{F})$ be the associated finite-element space as described in Section 1. By a mesh or a triangulation of $\mathbb{P}$ we shall mean the same thing, since we only consider conforming meshes.

We denote by $u_I = I_{\mathcal{T},\mathcal{F}}(u) \in S(\mathcal{T}, \mathcal{F})$ the interpolant of u. The interpolant $I_{\mathcal{T}}(u) = I_{\mathcal{T},\mathcal{F}}(u)$ has the following approximation property that generalizes classical results from [8, 18, 23, 42]. Many of the results below hold in greater dimensions, so we introduce d to be *the dimension of the domain*, assuming though that $d = 2$ for most of the results in this paper. In addition, note that when possible we consider the more general weighted Sobolev spaces, $\mathcal{W}_a^{m,p}$, and we indicate them by $\mathcal{K}^m$ when we assume that $p = 2$.

Recall that $|u|^p_{W^{b,p}(\mathbb{P})} := \sum_{|\alpha|=b} \|\partial^\alpha u\|^p_{L^p(\mathbb{P})}$ for $1 \le p < \infty$ and for $p = \infty$ we use the "max" value.

Theorem 2.1. *Fix $1 \le p \le \infty$. Also, let $\alpha > 0$ and $0 \le c < b \le m+1$ be fixed integers, with m as in equation (1.1), $b > s + d/p$ and $c \le s' + 1$. Then, there exists a constant $C(\alpha, m) > 0$ with the following property. Let $\mathcal{T}$ be a triangulation of $\mathbb{P}$ and assume that all triangles T in $\mathcal{T}$ have angles $\ge \alpha$ and sides of length $\le h$. Then, the interpolant $u_I := I_{\mathcal{T},\mathcal{F}}(u) \in S(\mathcal{T}, \mathcal{F})$ satisfies*

$$|u - u_I|_{W^{c,p}(\mathbb{P})} \le C(\alpha, m) h^{b-c} |u|_{W^{b,p}(\mathbb{P})},$$

for all $u \in W^{b,p}(\mathbb{P})$.

This result is well known for affine invariant families [18, 23]. The main point of this proof is to extend it to *conformally invariant* families of finite elements.

Proof. Consider a triangle, OAB, as shown in the left side of Figure 1 and denote by $\hat{O}$, $\hat{A}$, and $\hat{B}$ the measures of its angles. Denote by $\mathcal{S}$ the set of triangles, OAB, with the following properties:

(1) $O = (0, 0)$ is the origin,
(2) $A = (x, 0)$ is on the positive x-axis (that is, $x > 0$),

(3) the angles of the triangle satisfy $\hat{O} \leq \hat{A} \leq \hat{B}$ (so $|AB| \leq |OB| \leq |OA|$), and
(4) B is in the upper half-plane.

Then, any triangle, T, in the plane is congruent to a unique triangle, T_1 in $\mathcal{S}$. Denote the set of triangles, OAB in $\mathcal{S}$, for which $|OA| = x$, by $\mathcal{S}_x$. Since the set of conformal mappings contains the set of dilations, we have that every triangle, T, is conformally equivalent to a triangle, $T_1 \in \mathcal{S}_1$. The vertex, B, then completely determines the triangle, OAB in $\mathcal{S}_1$ (since O and A are fixed). Therefore, the set, $\mathcal{S}_1$, identifies with the set

$$\mathcal{B} := \{B = (x, y) \in \mathbb{R}^2,\ y > 0,\ x^2 + y^2 \leq 1,\ \text{ and } (x-1)^2 + y^2 \leq 1\}. \tag{2.1}$$

The set of triangles in $\mathcal{S}_1$ that have all angles $\geq \alpha$, hence, form a compact set, $K_\alpha \subset \mathcal{S}_1$. See the right side of Figure 1.

Since the range of each of the interpolants, I_T, contains the space, $\mathcal{P}_m$, of polynomials of degree $\leq m$ and $b \leq m+1$, we have that

$$|u - I_T(u)|_{W^{c,p}(T)} \leq C_T |u|_{W^{b,p}(\mathbb{R}^2)}, \tag{2.2}$$

by the Bramble-Hilbert Lemma [18,23]. We claim that the constant C_T in equation (2.2) can be chosen to be independent of $T \in K_\alpha \subset \mathcal{S}_1$ and, hence, to depend only on α and $c < b \leq m$.

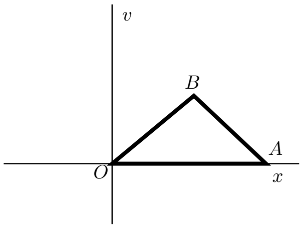
More precisely, assume that for all triangles T we always choose the best (smallest) value for C_T. Then, we claim that C_T is bounded on K_α. Indeed, this follows from the continuity of the family of interpolants, I_T, and the compactness of the set, K_α, as follows. Assume by contradiction that we cannot find a common constant bounding all the constants, C_T, with $T \in K_\alpha$. This means that for each n, we find a triangle, $T_n \in K_\alpha$, for which the best $C_{T_n} \geq n$. Since K_α is compact, we then find a convergent subsequence, $T_{n_k} \to T' \in K_\alpha$. Replacing T_n with this subsequence, we assume that $T_n \to T'$ (see the discussion before Definition 1.1 for the definition of convergence of triangles). Choose affine maps, L_n, such that $L_n(T_n) = T'$. Since $T_n \to T'$, we assume that $T_n \to I$, where I denotes the identity transformation. Also, the continuity property (Assumption 1) gives that $L_n(I_{T_n}) \to I_{T'}$ as $n \to \infty$ in the space of continuous linear maps, $\mathcal{L}(W^{b,p}(T'), \mathcal{P}_{m'})$. Denoting by $v = u \circ L_n^{-1}$ and noticing that T_n forms a bounded family of affine maps, we have that there exists a constant $C_b > 0$ independent of n, such that $|w|_{W^{c,p}(T_n)} \leq C_b |w \circ L_n^{-1}|_{W^{c,p}(T')}$ and $|v|_{W^{b,p}(T')} \leq C_b |u|_{W^{b,p}(T_n)}$ for all $w \in W^{c,p}(T_n)$ and $u \in W^{b,p}(T')$, which gives

$$\begin{aligned}
|u - I_{T_n}(u)|_{W^{c,p}(T_n)} &\leq C_b |u \circ L_n^{-1} - I_{T_n}(u) \circ L_n^{-1}|_{W^{c,p}(T')} \\
&= C_b |v - L_n(I_{T_n})(v)|_{W^{c,p}(T')} \\
&\leq C_b \big(|v - I_{T'}(v)|_{W^{c,p}(T')} + |I_{T'}(v) - L_n(I_{T_n})(v)|_{W^{c,p}(T')} \big) \\
&\leq C_b (C_{T'} + \epsilon) |v|_{W^{b,p}(T')} \leq C_b^2 (C_{T'} + \epsilon) |u|_{W^{b,p}(T_n)}\,,
\end{aligned}$$

for large n. Thus, $C_{T_n} \leq C_b^2(C_{T'} + \epsilon)$ for large n, which contradicts the fact that $C_{T_n} \to \infty$. (The constant C_b obtained from the fact that the family T_n is bounded satisfies, in fact, $C_b \to 1$ as $n \to \infty$.)

Next, if T is any triangle in $\mathcal{S}_x$, $x \leq h$ with all angles $\geq \alpha$, the dilation invariance of the interpolation, I_T, and of the seminorms, $|u|_{W^{c,p}(T)}$ and $|u|_{W^{b,p}(T)}$ (up to a factor), gives

$$|u - I_T(u)|_{W^{c,p}(T)} \leq C(\alpha, m) x^{b-c} |u|_{W^{b,p}(T)} \leq C(\alpha, m) h^{b-c} |u|_{W^{b,p}(T)}, \tag{2.3}$$



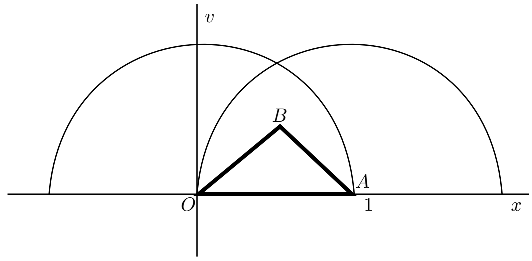


FIGURE 1. Left side: Sample triangle in $\mathcal{T}$. Right side: Depiction of the set, $\mathcal{S}_1$.

since $x = |OA|$ is the diameter of the triangle T. The invariance of all the terms of equation (2.3) under isometries, then, gives the same result for all triangles with angles $\leq \alpha$ and sides $\leq h$. Adding together the squares of all the estimates (2.3) for all the triangles in $\mathcal{T}$ gives the desired result. $\square$

We also obtain the following usual estimate.

Corollary 2.2. *Using the assumptions and notation of Theorem* 2.1*, there exists a constant $C = C(\alpha, m)$ such that*

$$\|u - u_I\|_{W^{c,p}(\mathbb{P})} \leq C(\alpha, m) h^{b-c} \|u\|_{W^{b,p}(\mathbb{P})}$$

for all $u \in W^{b,p}(\Omega)$.

Proof. For $p < \infty$, the proof follows by adding all the p powers of the estimates of Theorem 2.1 for (c, b) replaced by $(c-j, b-j)$, $j = 0, \ldots, c$. For $p = \infty$, we take the maximums. $\square$

The following estimate for the interpolation error on a proper subdomain of Ω (i.e., at a positive distance from the corners) then follows from the equivalence of the $W^{m,p}(\Omega)$-norm and the $\mathcal{W}_a^{m,p}(\Omega)$-norm on proper subsets of Ω. Recall that $r_\Omega(x)$ denotes the distance from x to the singular points of Ω, as in the introduction. If G is an open subset of Ω, define

$$\mathcal{W}_a^{m,p}(G; r_\Omega) := \{f : \Omega \to \mathbb{C},\ r_\Omega^{|\alpha|-a} \partial^\alpha f \in L^p(G), \text{ for all } |\alpha| \leq m\}, \tag{2.4}$$

and let $\|u\|_{\mathcal{W}_a^{m,p}(G)}$ denote the corresponding norm:

$$\|u\|^p_{\mathcal{W}_a^{m,p}(G)} = \sum_{|\alpha| \leq m} \|r_\Omega^{|\alpha|-a} \partial^\alpha f\|^p_{L^p(G)}. \tag{2.5}$$

(For $p = \infty$, we take the maximum.) Note that this definition is similar to that of the usual weighted Sobolev spaces $\mathcal{W}_a^m(\Omega)$ introduced in equation (0.2). In particular, $\mathcal{W}_a^{m,p}(\Omega) = \mathcal{W}_a^{m,p}(\Omega, r_\Omega)$. Also, we write $\mathcal{W}_a^{m,p}(G; r_\Omega) = \mathcal{W}_a^{m,p}(G)$ when convenient.

Proposition 2.3. *Let p, α, and $0 \leq c < b \leq m+1$ be as in Theorem* 2.1*. Also, let $G \subset \Omega$ be an open subset such that $r_\Omega > \xi > 0$ on G and let $\mathcal{T} = (T_j)$ be a triangulation of Ω with angles $\geq \alpha$ and sides $\leq h$. Then, for any given weights $a, a', \in \mathbb{R}$, there exists a constant $C = C(\alpha, m, \xi, a, a') > 0$ such that*

$$\|u - u_I\|_{\mathcal{W}_{a'}^{c,p}(G)} \leq C\, h^{b-c} \|u\|_{\mathcal{W}_a^{b,p}(G)}, \qquad \forall u \in \mathcal{W}_a^{b,p}(\Omega).$$

Proof. This follows from Theorem 2.1 and the equivalence of the $W^{t,p}$ and $\mathcal{W}_a^{t,p}$ norms on $\mathcal{W}_a^{t,p}(G; r_\Omega)$, for any t and a. Again, this equivalence holds due to the fact that G is bounded away from the singular points. Thus, the weights in the Sobolev norms are bounded from above and from below. □

Then, we have the following lemma, whose proof is a direct calculation.

Lemma 2.4. *There exist absolute constants C_m, $m \geq 0$, such that* $\|vu\|_{\mathcal{W}_a^{m,p}(G)} \leq C_m \|v\|_{\mathcal{W}_0^{m,\infty}(G)} \|u\|_{\mathcal{W}_a^{m,p}(G)}$.

2.2. **Graded κ-refinement.** The next step is to extend the above estimates of Proposition 2.3 to hold near the singular points. In [12], it has been shown that this can be done by considering graded meshes and the behavior of the spaces $\mathcal{K}_a^m$ under appropriate dilations. Most of the triangles in a graded mesh refinement (to be defined below) are divided into four equal triangles using the so-called *uniform refinement*.

Definition 2.5. Let T be a triangle, the *uniform refinement* of T is to decompose T as the union of four equal triangles using the midpoints of its sides. This is illustrated in Figure 2 if one takes A' and B' to be located at the midpoints of AQ and BQ, respectively, (i.e., $\kappa_Q = \frac{1}{2}$ using the notation of Definition 2.6) and with M being the midpoint of AB.

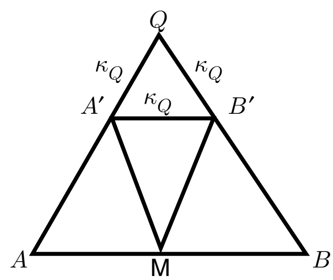


FIGURE 2. One refinement of the triangle T with singular point Q, and a given κ_Q. When $\kappa_Q = \frac{1}{2}$, we have uniform refinement.

The graded mesh refinement procedure depends on some choices of parameters. Thus, for each singular point Q of Ω, choose a number, $\kappa_Q \in (0, 1/2]$, and a set $\kappa(Q) = \{\kappa_Q^{(j)}\}$. A general procedure was developed in [12, 37] to give a refinement pattern that obtains optimal approximation properties. We extend this construction by considering more complicated refinements of certain trapezoids that arise in this construction. To this end, for each Q, in addition to κ_Q, choose

$$\kappa_Q = \kappa_Q^{(1)} < \kappa_Q^{(2)} < \ldots < \kappa_Q^{(j_Q)} = 1. \tag{2.6}$$

The refinement is described as follows. Again, assume that the term "singular point" refers to both geometric and artificial singular points described above. Recall that the initial mesh resolves the boundary exactly and that no edge of the initial mesh contains two singular points (in the sense just described).

Definition 2.6. Let $T = QAB$ be a triangle with a distinguished vertex Q. (In applications, Q will be a singular point of the problem.) Then, in a *κ-refinement* of T, first divide T into a smaller triangle $QA'B'$ with side lengths $QA' = \kappa_Q QA$ and $QB' = \kappa_Q QB$, and a trapezoid, $ABB'A'$ with AB parallel to $A'B' = \kappa_Q AB$, as shown in the left side of Figure 3. Then, refine the trapezoid $A'B'BA$ by dividing the segments QA and QB with the given ratios $\kappa_Q^{(i)}$ introduced in equation (2.6). More precisely, on AA', consider points $A_1, A_2, \ldots, A_{j_Q}$ such that $QA_i = \kappa_Q^{(i)} QA$ (in particular, $A_1 = A'$ and $A_{j_Q} = A$). Divide QB similarly and also consider the midpoint, M, of AB. Then, divide $ABB'A'$ into triangles by joining the corresponding points $A_1, A_2, \ldots, A_j$, $B_1, B_2, \ldots, B_j$, and using no point other than M. Thus, the resulting refinement of QAB into triangles uses only the points Q, A, B, M, and $A_1, A_2, \ldots, A_j$, $B_1, B_2, \ldots, B_j$.

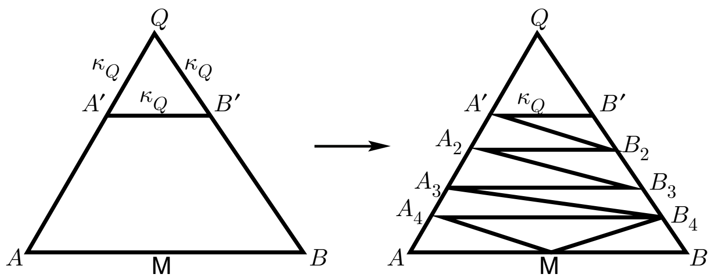


FIGURE 3. κ-refinement of triangle with singular point Q.

An example of a κ-refinement of a triangle $T = QAB$ is given in the right side of Figure 3, with $j = 5$. Note that if we apply the κ-refinement to two adjacent triangles that share the same distinguished point Q, then, the constants $\kappa_Q^{(i)}$ match, because they belong to the same singular point. This ensures that the resulting mesh refinement is conforming (i.e. there are no hanging nodes). See Figure 4. If further refinement is needed, the smaller triangle is refined with the same procedure described above. The trapezoid region is refined uniformly. This allows for further refinements to be done in a simple and recursive way.

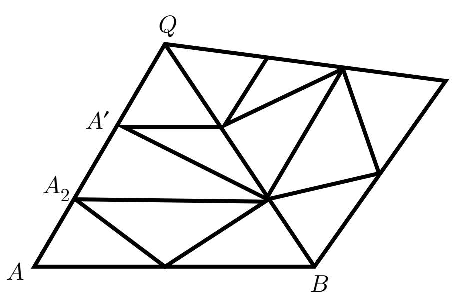


FIGURE 4. Matching κ-refinement for two triangles that share an edge and both touch the singular point Q.

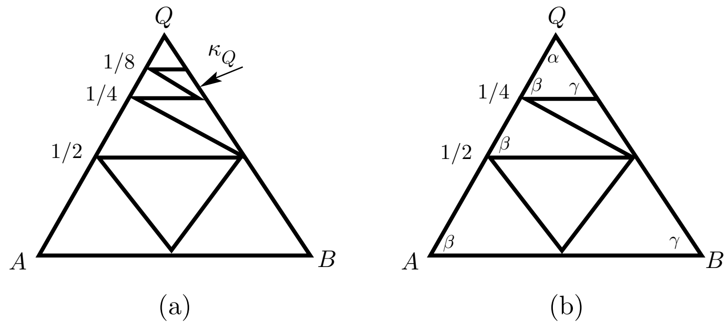


FIGURE 5. (a): κ-refinement of a triangle with singular point Q using $\kappa_Q \leq 1/8$ and $\kappa_Q^{(i)} = 2^{j_Q - i}$. (b): Similar to (a), but with one less bisection of the trapezoid. Angles are similar to angles in figure (a).

Two examples of κ-refinement of a triangle are given in Figure 5. Note that a variant of the division of the bottom trapezoid is compatible with the "newest vertex bisection" method [22].

Definition 2.7. Let $\mathcal{T}$ be a mesh such that every singular point of Ω is a vertex of a triangle in $\mathcal{T}$, no triangle of $\mathcal{T}$ contains more than one singular point from among the singular points of Ω, and the interface and the Dirichlet part of the boundary, $\partial_D\Omega$, is a union of edges of $\mathcal{T}$ (so $\mathcal{T}$ is aligned to the interface). A mesh with this property is called *admissible*. Then, define the *κ-refinement* of $\mathcal{T}$ to be the mesh $\kappa(\mathcal{T})$ obtained by applying uniform refinement to all triangles of T that contain no singular points and by applying the κ-refinement to all triangles T of $\mathcal{T}$ that contain a singular point of Ω (necessarily unique among the vertices of T). Then, the singular point of T will play the role of a distinguished point in the κ-refinement of T.

We then have the following simple observation.

Lemma 2.8. *With a fixed admissible initial mesh $\mathcal{T}_0$, $\kappa(\mathcal{T}_0)$ is also admissible and, hence, we can define by induction $\mathcal{T}_{n+1} = \kappa(\mathcal{T}_n)$.*

We also obtain the following proposition stating that the minimum angle of the meshes, $\mathcal{T}_n$, is bounded below from 0.

Proposition 2.9. *There exists an $\alpha_0 > 0$ that only depends on the angles of the initial mesh $\mathcal{T}_0$ and the constants $\kappa_Q^{(j)}$, such that the minimum angle in any triangle in the meshes, $\mathcal{T}_n$, satisfies*

$$\alpha_{\min} \geq \alpha_0.$$

Proof. This follows from the fact that all the triangles in the refinements, $\mathcal{T}_n$, belong to finitely many similarity classes, a fact easily proved by induction. $\square$

The following estimate on the number of triangles of $\mathcal{T}_n$ and of the dimension of the resulting finite element spaces $S_n = S(\mathcal{T}_n, \mathcal{F}_n)$ is useful for the optimal error estimates.

Proposition 2.10. *The number of triangles, $\#\mathcal{T}_n$, of the mesh $\mathcal{T}_n$ satisfies $\#\mathcal{T}_n \leq C2^{2n}$, for $C > 0$. Consequently, the dimension of the finite-element space, $S_n := S(\mathcal{T}_n, \mathcal{F})$, associated to the meshes $\mathcal{T}_n$, satisfies $\dim(S_n) \leq C2^{2n}$.*

Proof. The statement about S_n follows from the corresponding statement about $\mathcal{T}_n$. To prove the statement about the number of elements of $\mathcal{T}_n$, $\#\mathcal{T}_n$, we observe that each triangle of $\mathcal{T}_n$ is divided into four equal triangles in $\mathcal{T}_{n+1}$, unless that triangle contains a singular point, in which case it is divided into at most a triangles, where a is a fixed constant. Moreover, the number of triangles of $\mathcal{T}_n$ that contain a singular point is b, which is a constant independent of n. Let c_n be defined by $c_0 = \#\mathcal{T}_0$ and $c_{n+1} = (a-4)b + 4c_n$. Then,

$$c_n = 4^n \#\mathcal{T}_0 + (4^n - 1)(a-4)b/3 \leq C4^n.$$

Using, induction, this last inequality then gives

$$\#\mathcal{T}_{n+1} \leq ab + 4(\#\mathcal{T}_n - b) = (a-4)b + 4\#\mathcal{T}_n \leq c_{n+1} \leq C4^n,$$

which is the desired estimate. ☐

Since any triangle in the initial decomposition is divided into at least 4^n triangles in the n-level refinement mesh $\mathcal{T}_n$, we also have $2^{2n} \leq \dim(S_n)$, which, together with the inequality in Proposition 2.10 means

$$\dim(S_n) \sim 2^{2n}\,. \tag{2.7}$$

Assume that for each singular point, Q, a constant, $a_Q \in (0, 1]$, is given. (Also, recall that the set of singular points includes the vertices of Ω.) In applications, the constant a_Q comes from regularity estimates, but, in general, $\kappa_Q \leq 2^{-m/a_Q}$; see for instance the discussion in Section 3. The simplest method to perform the κ-refinement, in which each triangle is divided into four smaller triangles (i.e., $j_Q = 2$ for all Q as in Figure 2), leads to smaller and smaller angles as $a_Q \to 0$. (Incidentally, this simple refinement is related to the ones introduced in [2, 9, 12, 41] for the Dirichlet problem.) However, this is inconvenient in some applications and has disadvantages. Thus, we present a version of the κ-refinement procedure that leads to meshes with a lower bound on the minimum angle of the triangles in the refinements, independently of m, as follows.

The *uniform in m minimum angle preserving κ-refinement* is achieved by choosing in the definitions above $\kappa_Q^{(i)} = 2^{j_Q - i}$ with j_Q as small as possible, but satisfying $\kappa_Q := \kappa_Q^{(1)} \leq 2^{-m/a_Q}$. In particular, $\kappa_Q^{(1)} = 2^{j_Q-1} \leq 2^{-m/a_Q} < \kappa_Q^{(2)} = 2^{j_Q-2}$. We proceed similarly with BQ.

From a practical point of view, the uniform in m minimum angle preserving κ-refinement amounts to the side AQ being bisected until the shortest segment has length less than or equal to $2^{-m/a_Q}|AQ|$. The bottom trapezoid is refined into three triangles, whereas the intermediate trapezoids are merely bisected into two triangles. This is shown in the left side of Figure 5. This guarantees that the minimum angle of any triangle is bounded from below by an angle independent of m. We now prove that the version of the κ-refinement just introduced yields this result, which may be useful for the hp-version of the FEM.

Theorem 2.11. *Let the initial triangle ABQ be refined using the κ-refinement with $\kappa_Q^{(i)} = 2^{j_Q-i}$ and $\kappa_Q^{(1)} = \kappa_Q \leq 2^{-m/a_Q} < \kappa_Q^{(2)}$. Then, the minimum angle $\alpha_{\min}$ in* *any triangle in the meshes $\mathcal{T}_n$ is bounded from below,*

$$\alpha_{\min} \geq \alpha_0,$$

where α_0 only depends on the angles of the initial mesh $\mathcal{T}_0$ and is independent of m.

Proof. A simple geometric argument shows that the trapezoids constructed from this refinement are all similar. Therefore, the smallest angle obtained is reached if one were to bisect only one trapezoid. See the right side of Figure 5. The smallest angle in this configuration can be determined by the lengths of the sides and the angles of the initial triangle. Further refinement produces triangles with angles that are similar. Therefore, since m only determines the number of trapezoids introduced, the minimum angle is independent of this value. $\square$

The assumption that no two singular points of Ω belong to the same triangle of the mesh is not really needed. Any reasonable division of an initial triangulation will achieve this condition. For instance, if two singular points of Ω belong to the same triangle of the mesh, then the corresponding edge can be divided into equal parts or in a ratio given by the ratio of the corresponding κ constants. In this case, no new singular points are introduced on the sides of the initial triangle and this is much easier to implement. In extreme cases, however, large, skinny triangles with small angles could be introduced worsening the approximation results. In these instances, the more general procedure should be used instead. For instance, given a domain, D, with an initial triangulation with minimum angle α, the refinement can now be done in a way where there is a $\alpha_0 > 0$ (but dependent on κ and m) such that all the resulting meshes have the minimum angle greater than α_0 as shown in Theorem 2.11. In either case, the following definitions and approximation results hold.

2.3. **Approximation close to the singular points for function in $\mathcal{W}_a^{m,p}$.** Again, consider a mesh, $\mathcal{T}$, and a continuous, conformally invariant family, $\mathcal{F}$, of finite elements as in Subsection 1.2. We denote by $I_{\mathcal{T},\mathcal{F}}$ the interpolant associated to $\mathcal{T}$ and $\mathcal{F}$, as defined in Subsection 1.4. We denote by $\mathcal{T}_n$, the meshes on Ω, and by $\mathcal{F}$, the fixed conformal invariant family of finite elements. Then, $I_{\mathcal{T}_n,\mathcal{F}}$ denotes the associated interpolating operator.

We now want to investigate the approximation properties afforded by the triangulation $\mathcal{T}_n$ close to a singular point, Q, of Ω for functions $u \in \mathcal{W}_a^{m,p}(\Omega)$. We again consider the general case of $1 < p < \infty$ and equations of order 2μ. Approximation results for functions in $\mathcal{W}_a^{m,p}$ are probably enough to treat problems with only Dirichlet boundary conditions and no interfaces. When interfaces or other types of boundary conditions are present, one most likely would need a generalization of this framework to use augmented Sobolev spaces. An example of how this is done for $\mu = 1$ can be seen in the following subsection.

Denote V_Q to be the union of the (closed) triangles in the initial mesh $\mathcal{T}_0$ that have Q as a singular point. Denote by $|x - y|$ the Euclidean distance from x to y and assume that $r_\Omega(x) = |x - Q|$ for $x \in V_Q$. By refining the initial mesh, if necessary, we assume the closed sets V_Q are disjoint.

For any region $G \subset V_Q$, denote by $\lambda G \subset V_Q$ the region obtained by dilating G with respect to Q with ratio $\lambda < 1$. The following interpolation estimates are then similar to those in [12, 37], but deal with a higher order approximation. In particular, we need the following simple lemma that for $p = 2$ is proved by a direct calculation in [12]. Recall the definitions of the norms $\| \; \|_{\mathcal{W}_a^{m,p}(G)}$ from equation (2.5). The slightly more general lemma below is also proved by a direct calculation (recall that $d = 2$ in this paper).

Lemma 2.12. *Let Q be a singular point of Ω and $G \subset V_Q \subset \overline{\Omega} \subset \mathbb{R}^d$ an open set. Denote $G' = \lambda G$, $0 < \lambda < 1$, and $u_\lambda(x) := u(\lambda x)$, then*

$$\|u_\lambda\|_{\mathcal{W}_a^{m,p}(G)} = \lambda^{a-d/p} \|u\|_{\mathcal{W}_a^{m,p}(G')} .$$

The condition $\lambda < 1$ is not necessary as long as the domains of the resulting functions are mapped appropriately.

Lemma 2.13. *Assume $p > 1$ and $\epsilon > 0$. Then, there is a continuous embedding $\mathcal{W}_{\epsilon+d/p}^{m+\mu,p}(\Omega) \to \mathcal{C}(\overline{\Omega})$. Moreover, for each singular point $Q \in \mathcal{V}$ and for each $u \in \mathcal{W}_{\epsilon+d/p}^{m+\mu,p}(\Omega)$, the value $u(Q) = 0$ is well defined.*

Proof. Let $u \in V$ and $U \subset \Omega$ be an open subset at positive distance from the set of singular points $\mathcal{V}$. Then, the Sobolev embedding theorem gives $W^{m+\mu,p}(U) \subset \mathcal{C}(\overline{U})$, since $p^{-1} < 1 \le (m+\mu)/d$ (where, we recall, $d = 2$ is the dimension of the domain). Hence, $u \in \mathcal{C}(\overline{U})$. By the dilation invariance of the spaces $\mathcal{W}_a^{m,p}(\Omega_j)$, Lemma 2.12, we obtain

$$\mathcal{W}_{d/p}^{m+\mu,p}(\Omega_j) \subset \mathcal{C}(\overline{\Omega} \smallsetminus \mathcal{V}) \cap L^\infty(\Omega).$$

In the case that there are no interfaces, this gives

$$\mathcal{W}_{\epsilon+d/p}^{m+\mu,p}(\Omega) = r_\Omega^\epsilon \mathcal{W}_{d/p}^{m+\mu,p}(\Omega) \subset \mathcal{C}(\overline{\Omega}) ,$$

since $\epsilon > 0$. Finally, $u(Q) = 0$, since $r_\Omega^\epsilon(Q) = 0$. ☐

Assumptions. *For simplicity of the notation,* assume from now on that the constants κ_Q are all the same and let $\kappa = \kappa_Q$. Also assume that $\kappa \le 2^{-m/a}$, for some fixed $a > 0$. *Recall that the approximation degree $m \ge 1$ is fixed.*

Now estimate the interpolation error on the triangles of the mesh $\mathcal{T}_n$ that are close to a singular point. More precisely, for each singular point $Q \in \mathcal{V}$, define $V_Q^{(n)}$ to be the union of all the closures of triangles $T \in \mathcal{T}_n$ that have Q as a vertex. Similarly, define $U_Q^{(n)}$ to be the union of all the closures of triangles $T \in \mathcal{T}_n$ that touch (i.e., intersect) $V_Q^{(n)}$. Then, choose functions $\eta_Q^{(n)} \in \mathcal{C}^\infty(\overline{\Omega})$ that are equal to 0 on $V_Q^{(n)}$ and are equal to 1 outside $U_Q^{(n)}$. These functions, $\eta_Q^{(n)}$, can be chosen such that they correspond to each other with respect to dilations centered at Q, in an obvious sense. Thus, for each n, $\eta_Q^{(n)}$ is obtained by a dilation with ratio κ^{n-1} from $\eta_Q^{(1)}$.

Lemma 2.14. *Assume $a > 0$ and $\kappa \le 2^{-m/a}$ and let $G \subset \Omega$ be an open subset. Using the functions $\eta_Q^{(n)}$ introduced above, there exists a constant $C > 0$, independent of n but possibly dependent on a, b, m, p, and μ, such that*

$$\|(1 - \eta_Q^{(n)})u\|_{\mathcal{W}_{b+d/p}^{\mu,p}(G)} \le C 2^{-mn} \|u\|_{\mathcal{W}_{a+b+d/p}^{m+\mu,p}(G)},$$

for any $u \in \mathcal{W}_{a+b+d/p}^{m+\mu,p}(G, r_\Omega)$.

Proof. Assume $G = \Omega$, for simplicity of notation. The general case is identical. The support of $(1-\eta_Q^{(n)})u$ is contained in a set on which $r_\Omega \le C\kappa^n$, with C independent of Q or n and, hence,

$$\|(1-\eta_Q^{(n)})u\|_{\mathcal{W}^{\mu,p}_{b+d/p}} \le C(\kappa^n)^a \|(1-\eta_Q^{(n)})u\|_{\mathcal{W}^{\mu,p}_{a+b+d/p}} \le C2^{-mn}\|u\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}}, \tag{2.8}$$

where in the last inequality, Lemma 2.4 is used as well as the fact that the norms $\|1-\eta_Q^{(n)}\|_{\mathcal{W}^{m,\infty}_0}$ are independent of n by the dilation invariance of the $\mathcal{W}^{m,\infty}_0$ norm (Lemma 2.12). ☐

We assume from now on that we have fixed constants a and b such that $a > 0$ and $a + b > 0$. We also assume that $\kappa \le 2^{m/a}$.

To continue the study of the error estimates, notice that we may assume that by refining the initial mesh $\mathcal{T}_0$, all the functions $1-\eta_Q^{(n)}$ have disjoint support, so $(1-\eta_Q^{(n)})(1-\eta_{Q'}^{(n)}) = 0$ for $Q \ne Q' \in \mathcal{V}$ and, hence, $\eta^{(n)} := \prod_{Q\in\mathcal{V}}\eta_Q^{(n)}$ satisfies $1 = \eta^{(n)} + \sum_Q(1-\eta_Q^{(n)})$. Denote then by

$$\tilde{u}_n = \eta^{(n)}u + \sum_{Q\in\mathcal{V}} u(Q)(1-\eta_Q^{(n)}) = \eta^{(n)}u\,, \tag{2.9}$$

for $u \in \mathcal{W}^{m+\mu,p}_{a+b+d/p}(\Omega)$. Note that $\tilde{u}_n$ is well defined since V consists of continuous functions (Lemma 2.13) by the assumption that $a + b > 0$.

Lemma 2.14 then yields the following corollary.

Corollary 2.15. *Using the notation $\tilde{u}_n$ of equation* (2.9)*, we have that there exists a constant $C > 0$, independent of n, such that*

$$\|u-\tilde{u}_n\|_{\mathcal{W}^{\mu,p}_{b+d/p}(G)} \le C2^{-mn}\|u\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(G)},$$

for any $u \in \mathcal{W}^{m+\mu,p}_{a+b+d/p}(G)$, $G \subset \Omega$ open.

Proof. Since $1 = \eta^{(n)} + \sum_{Q\in\mathcal{V}}(1-\eta_Q^{(n)})$, we have

$$u - \tilde{u}_n = \sum_{Q\in\mathcal{V}} u(1-\eta_Q^{(n)}).$$

The result then follows from Lemma 2.14. ☐

Next, we introduce the *modified interpolation operator* by

$$u_{I,n} = I_{\mathcal{T}_n,\mathcal{F}}(\tilde{u}_n) \in S_n := S(\mathcal{T}_n, \mathcal{F}), \tag{2.10}$$

with $\tilde{u}_n$ defined in equation (2.9). Note that the modified interpolation operator allows the smoothness properties of u close to the singular points to be ignored, since $\tilde{u}_n$ (unlike u) is constant close to each singular point. Thus, only smoothness estimates on u away from the singular points are needed, where they are the same as the classical estimates. An example is the Sobolev embedding $H^2 \subset \mathcal{C}$, valid in two and three dimensions.

Recall that we want to approximate functions u in $\mathcal{W}^{m+\mu,p}_{a+b+d/p}(\Omega)$, $a > 0$, $a + b > 0$ with functions in S_n. We in fact show that the interpolant $u_{I,n}$ is a good approximation of u. The estimates are obtained by breaking them into regions. We begin with the regions closest to the singularities.

Lemma 2.16. *Denote by $\kappa^n T \subset T \subset \Omega_j$ the triangle with singular point, Q, obtained from $T \in \mathcal{T}_0$ after n refinements. Let $u \in \mathcal{W}^{m+\mu,p}_{a+b+d/p}(\Omega_j)$. Then,*

$$\|u - u_{I,n}\|_{\mathcal{W}^{\mu,p}_{b+d/p}(\kappa^n T)} \leq C 2^{-mn} \|u\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(\kappa^n T)},$$

where C depends on m and κ, but not on n or T. Here, $u_{I,n}$ is the modified interpolant given by equation (2.10)*.*

Proof. Since $\tilde{u}_n = u(Q) = 0$ and $\eta^{(n)} = 0$ on $\kappa^n T$, we have $u_{I,n} = I_{\mathcal{T}_n,\mathcal{F}}(\tilde{u}_n) = u(Q) = 0$ on $\kappa^n T$ as well. Then,

$$\|u - u_{I,n}\|_{\mathcal{W}^{\mu,p}_{b+d/p}(\kappa^n T)} = \|u\|_{\mathcal{W}^{\mu,p}_{b+d/p}(\kappa^n T)} \leq C 2^{-mn} \|u\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(\kappa^n T)},$$

by Lemma 2.14. □

The bounds on $\kappa^n T$ of the previous lemma are combined with bounds on sets of the form $\kappa^j T \smallsetminus \kappa^{j+1} T$ to obtain the following estimate on the arbitrary, but fixed, triangle $T \in \mathcal{T}_0$ that has a vertex Q that is a singular point of Ω. Recall that λG is obtained from G by dilating with ratio $\lambda < 1$ and center Q.

Proposition 2.17. *Consider the triangles $\kappa\xi T \subset \xi T \subset \Omega_j$, where T is a triangle with one vertex $Q \in \mathcal{V}$, a singular point of Ω. Let $\mathcal{T} = (T_j)$ be a triangulation of $G := \xi T \smallsetminus \kappa\xi T$ with angles $\geq \alpha$ and edges $\leq h$. Let $u \in \mathcal{W}^{m+\mu,p}_{a+b+d/p}(\Omega_j)$. Then, the interpolant $I_{\mathcal{T},\mathcal{F}}(u) \in S(\mathcal{T},\mathcal{F})$ satisfies*

$$\|u - I_{\mathcal{T},\mathcal{F}}(u)\|_{\mathcal{W}^{\mu,p}_{b+d/p}(G; r_\Omega)} \leq C(a,\kappa,\alpha,m)\xi^a (h/\xi)^m \|u\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(G; r_\Omega)},$$

with $C(a,\kappa,\alpha,m)$ independent of ξ, h, and u.

Proof. Let $\tilde{\ell}$ be the distance from Q to the opposite side of T. Assume that Q is the origin, to simplify the notation, and recall the dilation function u_λ, where $u_\lambda(x) = u(\lambda x)$, $x \in \mathbb{R}^2$. Also, recall that the dilation commutes with interpolation by the assumption that the family of finite elements, $\mathcal{F}$, is conformally invariant. Using Lemma 2.12 with $\lambda = \xi$, Proposition 2.3 is applied to the region $G' = T \smallsetminus \kappa T = \lambda^{-1} G \subset \Omega$. Denoting by $M = C(\alpha, m, \kappa\tilde{\ell}\xi, a + b + d/p, b + d/p)$ the constant of Proposition 2.3, we obtain

$$\begin{aligned}
\|u - I_{\mathcal{T},\mathcal{F}}(u)\|_{\mathcal{W}^{\mu,p}_{b+d/p}(G)} &= \lambda^{-b} \|u_\lambda - I_{\mathcal{T},\mathcal{F}}(u)_\lambda\|_{\mathcal{W}^{\mu,p}_{b+d/p}(G')} \\
&= \lambda^{-b} \|u_\lambda - I_{\mathcal{T},\mathcal{F}}(u_\lambda)\|_{\mathcal{W}^{\mu,p}_{b+d/p}(G')} \leq M \lambda^{-b} (h/\lambda)^m \|u_\lambda\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(G')} \\
&= M \lambda^{-b} \lambda^{a+b} (h/\lambda)^m \|u\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(G)} = M \xi^a (h/\xi)^m \|u\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(G)}.
\end{aligned}$$

This completes the proof. □

Next, denote $u_{I,n}$ to be the modified interpolant of equation (2.10).

Proposition 2.18. *Let $T \in \mathcal{T}_0$ have the singular point Q as a vertex. Then, there exists a constant $C > 0$, such that*

$$\|u - u_{I,n}\|_{\mathcal{W}^{1,p}_{b+1}(T)} \leq C 2^{-mn} \|u\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(\Omega)},$$

for all n and all $u \in \mathcal{W}^{m+\mu,p}_{a+b+d/p}(\Omega)$.

Proof. Fix n. The proof of the proposition follows from the estimates on the subsets $\kappa^{j-1}T \smallsetminus \kappa^j T$, $1 \le j \le n$, (Proposition 2.17) and from the estimate on $\kappa^n T$ (Lemma 2.16). Let $\tilde{u}_n$ be as defined in equation (2.9). Recall that we write $\mathcal{W}_a^{m,p}(G; r_\Omega) = \mathcal{W}_a^{m,p}(G)$ when convenient. In view of Corollary 2.15, it is enough to show that

$$\|\tilde{u}_n - u_{I,n}\|_{\mathcal{W}^{\mu,p}_{b+d/p}(T)} \le C 2^{-mn} \|u\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(T)} ,$$

with C a possibly different constant. Then, write

$$\|\tilde{u}_n - u_{I,n}\|^p_{\mathcal{W}^{\mu,p}_{b+d/p}(T)} = \|\tilde{u}_n - u_{I,n}\|^p_{\mathcal{W}^{\mu,p}_{b+d/p}(\kappa^n T)} + \sum_{j=1}^{n} \|\tilde{u}_n - u_{I,n}\|^p_{\mathcal{W}^{\mu,p}_{b+d/p}(\kappa^{j-1}T \smallsetminus \kappa^j)} .$$

Recall that $u_{I,n} = I_{\mathcal{T}_n,\mathcal{F}}(\tilde{u}_n) \in S_n := S(\mathcal{T}_n, \mathcal{F})$ and that $\tilde{u}_n = 0$ on $\kappa^n T$. The first term $\|\tilde{u}_n - u_{I,n}\|^p_{\mathcal{W}^{\mu,p}_{b+d/p}(\kappa^n T)}$ is, thus, zero. Definition 2.7 shows that the mesh size h of the restriction of the mesh $\mathcal{T}_n$ to $\kappa^{j-1}T \smallsetminus \kappa^j T$, is $\le C\kappa^{j-1}2^{j-1-n}$, for a constant C that depends only on $\mathcal{T}_0$. Let $G = \kappa^{j-1}T \smallsetminus \kappa^j T$. Then, using the notation in Proposition 2.17, we have that $\xi = \kappa^{j-1}$ and, therefore,

$$\begin{aligned} \|\tilde{u}_n - u_{I,n}\|_{\mathcal{W}^{\mu,p}_{b+d/p}(G)} &\le C_1 \kappa^{(j-1)a} (\kappa^{j-1} 2^{j-1-n}/\kappa^{j-1})^m \|\tilde{u}_n\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(G)} \\ &\le C_2 2^{-(j-1)m} 2^{-nm+(j-1)m} \|\tilde{u}_n\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(G)} = C_2 2^{-nm} \|\tilde{u}_n\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(G)} \\ &\le C_3 2^{-nm} \|\tilde{u}_n\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(G)}, \end{aligned} \tag{2.11}$$

where C_1, C_2, and C_3 depend on κ, but not on u, n, or j. The last inequality is from Lemma 2.14. For $p < \infty$, we complete the proof by adding up all the above error estimates on the subsets $G := \kappa^{j-1}T \smallsetminus \kappa^j T$, $1 \le j \le n$. For $p = \infty$, we simply take the maximum. ☐

We then have the following main approximation result.

Theorem 2.19. *Assume $u \in \mathcal{W}^{m+\mu,p}_{a+b+d/p}(\Omega)$, $a > 0$, $a + b > 0$, and $\kappa \le 2^{-m/a}$. Let $\mathcal{T}_n$ be the n-th refinement of an initial triangulation, $\mathcal{T}_0$. Let $S_n := S_n(\mathcal{T}_n, \mathcal{F})$ be the associated finite-element space given in equation* (1.8) *and let $u_{I,n} \in S_n$ be the modified interpolant associated to $\mathcal{T}_n$ and $\mathcal{F}$, equation* (2.10)*. Then, there exists $C > 0$, independent of n or u, such that*

$$\|u - u_{I,n}\|_{\mathcal{W}^{\mu,p}_{b+d/p}(\Omega)} \le C 2^{-mn} \|u\|_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(\Omega)} .$$

Proof. Consider now the set $W = \Omega \smallsetminus \left(\bigcup_{Q \in \mathcal{V}} \bigcup_{Q \in T \in \mathcal{T}_0} \overline{T}\right)$, that is, the set G is obtained by removing from Ω all the triangles of the initial mesh $\mathcal{T}_0$ that have a vertex among the singular points of the problem. (We write by abuse of notation, $Q \in T$, when we really mean that the vertex Q of T is in the closure of T.)

Assume first that $p < \infty$. The proof is an immediate consequence of the estimates in Propositions 2.24 and 2.3 applied, respectively, to the triangles $T \in \mathcal{T}_0$ that have singular points as vertices and to the region W that is the complement of these triangles in Ω (as defined above). We then have

$$\begin{aligned}\|u-u_{I,n}\|^p_{\mathcal{W}^{\mu,p}_{b+d/p}(\Omega)} &= \sum_{Q\in\mathcal{V}}\sum_{Q\in T}\|u-u_{I,n}\|^p_{\mathcal{W}^{\mu,p}_{b+d/p}(T)}+\|u-u_{I,n}\|^p_{\mathcal{W}^{\mu,p}_{b+d/p}(W)}\\ &\le C2^{-mnp}\big(\sum_{Q\in\mathcal{V}}\sum_{Q\in T}\|u\|^p_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(T)}+\|u\|^p_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(W)}\big)\\ &\le C2^{-mnp}\|u\|^p_{\mathcal{W}^{m+\mu,p}_{a+b+d/p}(\Omega)}.\end{aligned}$$

For $p=\infty$, we take the maximum. The proof is now complete. ☐

2.4. **Extension to some transmission problems.** Let us again assume that $p=2$ and $\mu=1$, but now allow for transmission problems and mixed boundary conditions. In this case, we need to use augmented broken Sobolev spaces as in [37] and recall the definition of these spaces. Let χ_Q be a smooth function that is equal to 1 near each singular point $Q\in\mathcal{V}$. We assume that the functions, χ_Q, have disjoint supports and, in case Q is an interior point of the domain, then χ_Q vanishes close to the boundary of Ω. If χ_Q is supported near a vertex of Ω, then we assume that it satisfies the Neumann boundary conditions on each adjacent side. If the diffusion matrix, A, is scalar, then we can take χ_Q to be a function of the distance function to the point Q. Denote by $\mathcal{U}\subset\mathcal{V}$, the subset of the set $\mathcal{V}$ of singular points of the problem that are either a vertex whose adjacent sides have Neumann conditions (a so-called "Neumann-Neumann vertex"), or a nonsmooth interface point interior to Ω, or a point where the interface touches the interior of $\partial_N\Omega$. Thus, $\mathcal{U}$ includes the points that belong to more than two of the subdomains, $\overline{\Omega}_j$ (the so-called *multiple junction points*), which are typically interior points of Ω. We also have that $\mathcal{U}$ is the set of points $Q\in\mathcal{V}$ such that χ_Q satisfies the boundary conditions of the problem. Then, we define U_s to be the linear span of the functions χ_Q, and, hence, all the functions in U_s satisfy the boundary conditions of the problem. We also need to introduce the *broken Sobolev spaces* $\hat{\mathcal{K}}^m_a(\Omega)$, defined in terms of the decomposition $\overline{\Omega}=\bigcup_{j=1}^K\overline{\Omega}_j$,

$$\hat{\mathcal{K}}^m_a(\Omega):=\{u:\Omega\to\mathbb{R},\ u|_{\Omega_j}\in\mathcal{K}^m_a(\Omega_j),\ j=1,\dots,K\}. \tag{2.12}$$

We then let the approximation space be the following augmented broken Sobolev space:

$$V\ :=\ \Big(\hat{\mathcal{K}}^{m+1}_{a+b+1}(\Omega)\cap\mathcal{K}^1_{a+b+1}(\Omega)\Big)+U_s, \tag{2.13}$$

for the *fixed* approximation parameter $m\ge 1$ and some fixed parameters $a>0$, $a+b>0$ as in the previous section. This choice of approximation space is suggested by the regularity results of [37], which state that the solution, u, of the transmission/boundary value problem (0.1) is such that $u\in V$ for $b=0$ (see [36] for some related results). The additional parameter b satisfying $a+b>0$ is introduced with some applications to the least squares finite-element method in mind [34]. Notice that for each triangle $T\in\mathcal{T}_n$, we have $T\subset\Omega_j$, for some j, since the initial mesh $\mathcal{T}_0$ is aligned with the interface and, hence, $u\in\mathcal{K}^{m+1}_{a+b+1}(T)$. Thus, we work with the broken weighted Sobolev spaces in the same way we would work with the usual weighted Sobolev spaces. Notice that the notation in the approximation results is not optimized (it would have been maybe easier to use $b=0$), however, the form of this notation is the one that will be used in some of the later applications.

The norm on V is given by the direct sum norm for any choice of a norm $\| \; \|_{U_s}$ on the finite-dimensional space U_s:

$$\|u_0 + w_s\|_V = \|u_0\|_{\mathcal{K}^1_{a+b+1}(\Omega)} + \sum_{j=1}^{K} \|u_0\|_{\mathcal{K}^{m+1}_{a+b+1}(\Omega_j)} + \|w_s\|_{U_s}, \tag{2.14}$$

where $u_0 \in \hat{\mathcal{K}}^{m+1}_{a+b+1}(\Omega) \cap \mathcal{K}^1_{a+b+1}(\Omega)$ and where w_s is always an element in U_s. Then, we have the following lemma (which remains valid in three dimensions). For definiteness, we choose $\| \sum_Q a_Q \chi_Q \|_{U_s} = \sum_Q |a_Q|$. Also, notice that $\mathcal{K}^1_{a+b+1}(\Omega) \cap \hat{\mathcal{K}}^{m+1}_{a+b+1}(\Omega)$ is a closed subspace of $\hat{\mathcal{K}}^{m+1}_{a+b+1}(\Omega)$ for $m \geq 1$, and, hence, the term $\|u_0\|_{\mathcal{K}^1_{a+b+1}(\Omega)}$ is not really necessary in the definition of the V-norm. Another good choice of a norm on U_s is the restriction of any Sobolev norm H^k on Ω to U_s.

We want to extend Theorem 2.19 to $u \in V$. Our reasoning parallels the one in the previous section.

The following lemma relies essentially on the additional condition that $u_0 \in \mathcal{K}^1_{a+b+1}(\Omega)$, which enforces the continuity across the interface(s).

Lemma 2.20. *There is a continuous embedding $V \to \mathcal{C}(\overline{\Omega})$. In particular, for each singular point $Q \in \mathcal{V}$ and for each $u \in V$, the value $u(Q)$ is well defined and depends continuously on u.*

Proof. Let $u = u_0 + w_s \in V$. We have $w_s \in U_s \subset \mathcal{C}(\overline{\Omega})$, so we assume $u = u_0 \in \hat{\mathcal{K}}^{m+1}_{a+b+1}(\Omega)$. The same argument as in the proof of Lemma 2.13 then gives that $u \in \mathcal{C}(\overline{\Omega_j})$. The continuity across the interfaces, and hence the global continuity follows from $u \in \mathcal{K}^1_{a+b+1}(\Omega)$. □

It follows that any $u \in V := \Big(\hat{\mathcal{K}}^{m+1}_{a+b+1}(\Omega) \cap \mathcal{K}^1_{a+b+1}(\Omega)\Big) + U_s$ can be written as

$$\begin{gathered} u = u_0 + w_s, \quad \text{where} \quad w_s := \sum_{Q \in \mathcal{U}} u(Q)\chi_Q \in U_s \quad \text{and} \\ u_0 \in \hat{\mathcal{K}}^{m+1}_{a+b+1}(\Omega) \cap \mathcal{K}^1_{a+b+1}(\Omega)\,, \end{gathered} \tag{2.15}$$

and we call this decomposition the *canonical decomposition* of $u \in V$.

Recall the functions $\eta_Q^{(n)}$ and $\eta^{(n)} := \prod_{Q \in \mathcal{V}} \eta_Q^{(n)}$ introduced in the previous subsection before equation (2.9). Denote then by

$$\tilde{u}_n = \eta^{(n)} u + \sum_{Q \in \mathcal{V}} u(Q)(1 - \eta_Q^{(n)}) = \eta^{(n)} u + \sum_{Q \in \mathcal{U}} u(Q)(1 - \eta_Q^{(n)}), \tag{2.16}$$

for $u \in V$. Note that $\tilde{u}_n$ is well defined since V consists of continuous functions (Lemma 2.20). Of course, when there are no interfaces and no mixed boundary conditions, we have that $\mathcal{U}$ is empty, and hence $\tilde{u}_n = \eta^{(n)} u$.

Lemma 2.14 then yields the following corollaries.

Corollary 2.21. *Using the notation $\tilde{u}_n$ of equation* (2.16)*, there exists a constant $C > 0$, independent of n, such that*

$$\|u - \tilde{u}_n\|_{\mathcal{K}^1_{b+1}(\Omega)} \leq C 2^{-mn} \|u\|_V,$$

for any $u \in V$. (So u has the canonical decomposition $u = u_0 + w_s$, $w_s \in U_s$ and $u_0 \in \hat{\mathcal{K}}^{m+1}_{a+b+1}(\Omega) \cap \mathcal{K}^1_{a+b+1}(\Omega)$.)

Proof. The proof is obtained by estimating the error term $\|u-\tilde{u}_n\|_{\mathcal{K}^1_{b+1}(\Omega)}$ separately on each region Ω_j, so that $K = 1$ may be assumed (that is, that there are no interfaces). Then $u = u_0 + \sum_Q u(Q)\chi_Q$, with $u_0 \in \mathcal{K}^{m+1}_{a+b+1}(\Omega) = \hat{\mathcal{K}}^{m+1}_{a+b+1}(\Omega)$, by the definition of the approximation space V. Since $1 = \eta^{(n)} + \sum_Q(1-\eta^{(n)}_Q)$, we have

$$u - \tilde{u}_n = \sum_{Q\in\mathcal{U}} (u - u(Q))(1-\eta^{(n)}_Q). \tag{2.17}$$

Since $(1-\chi_Q)(1-\eta^{(n)}_Q)$ is a smooth function on $\overline{\Omega}$ that is zero near the set $\mathcal{V}$ of singular points, we have $(1-\chi_Q)(1-\eta^{(n)}_Q) \in \mathcal{K}^{m+1}_{a+b+1}(\Omega)$ for all m, a, and b.

By decreasing supports of the functions $\eta^{(n)}_Q$, if necessary, assume that $\chi_{Q'}(1-\eta^{(n)}_Q) = 0$ for $Q \neq Q'$ and that $(1-\chi_Q)(1-\eta^{(n)}_Q) = 0$. Write $u \in V$ as $u = u_0 + \sum_{Q\in\mathcal{U}} u(Q)\chi_Q$, with $u_0 \in \mathcal{K}^{m+1}_{a+b+1}(\Omega)$. Equation (2.17) then gives

$$u - \tilde{u}_n = \sum_{Q\in\mathcal{U}} u_0(1-\eta^{(n)}_Q). \tag{2.18}$$

The result then follows from Lemma 2.14. □

We also mention the following corollary of the above proof.

Corollary 2.22. *Using the notation of Corollary* 2.21*, if $u \in U_s$, then $u = \tilde{u}_n$.*

Proof. Indeed, under the assumptions of this corollary, $u_0 = 0$ and, hence, the result is given by equation (2.18). □

Next, we show that we may assume $u \in \mathcal{K}^{m+1}_{a+b+1}(\Omega)$ (that is, we may take $w_s = 0$). Let $T \in \mathcal{T}_0$ be a triangle that has a vertex that is a singular point Q of Ω. Let $\kappa^n T \in \mathcal{T}_n$ be the triangle obtained by dilating T with ratio $\kappa^n = \kappa^n_Q$ and center Q, that is, the triangle that is similar to T with ratio κ^n, has Q as a vertex, and has all sides parallel to the sides of T. Then, $\kappa^n T \subset \kappa^{n-1}T$ for $n \geq 1$ and $\kappa^n T \in \mathcal{T}_n$. We assume that for all triangles $T \in \mathcal{T}_0$ with singular vertex $Q \in \mathcal{V}$, we have $\chi_Q = 1$ on κT, for all Q.

Lemma 2.23. *Let $w = \sum_Q a_Q\chi_Q \in U_s$. Then,*

$$\|w - w_{I,n}\|_{\mathcal{K}^1_{b+1}(\Omega)} \leq C2^{-mn}\sum_Q |a_Q| =: C2^{-mn}\|w\|_{U_s} =: C2^{-mn}\|w\|_V,$$

for a constant C that is independent of w and n.

Proof. Let $U = \Omega \smallsetminus \bigcup \kappa T$, where the union is over all triangles $T \in \mathcal{T}_0$ that have a vertex in the set $\mathcal{V}$ of singular points of the problem. Then, $w = w_{I,n}$ outside U, by Corollary 2.22. The result is then a consequence of Proposition 2.3 and of the fact that $h \leq C2^{-n}$ for the mesh $\mathcal{T}_n$, with C a constant depending only on the initial mesh $\mathcal{T}_0$. □

We want to extend the above lemma to $u \in V$. By linearity, we assume that $u \in \mathcal{K}^{m+1}_{a+b+1}(\Omega)$, using the above lemma.

Proposition 2.24. *Let $T \in \mathcal{T}_0$ have the singular point Q as a vertex. Then, there exists a constant $C > 0$, such that*

$$\|u - u_{I,n}\|_{\mathcal{K}^1_{b+1}(T)} \le C2^{-mn}\|u\|_V = C2^{-mn}\big(\|u_0\|_{\mathcal{K}^{m+1}_{a+b+1}(T)} + \|w_s\|_{U_s}\big),$$

for all n and all $u \in V$ with canonical decomposition $u = u_0 + w_s$.

Proof. We assume that $w_s = 0$, by Lemma 2.23. The result then follows from the analogous Proposition 2.18. □

Now we state one of the main results that is used in the application to the numerical approximation of the solution, u, of problem (0.1).

Theorem 2.25. *Assume $u = u_0 + w_s \in V$, where $u_0|_{\Omega_j} \in \mathcal{K}^{m+1}_{a+b+1}(\Omega_j)$ for all j, $u_0 \in \mathcal{K}^1_{a+b+1}(\Omega)$, and $w_s = \sum_{Q\in\mathcal{U}} a_Q\chi_Q \in U_s$. Assume $a > 0$, $a + b > 0$, and $\kappa \le 2^{-m/a}$. Let $\mathcal{T}_n$ be the n-th refinement of an initial triangulation, $\mathcal{T}_0$, aligned with the interface. Let $S_n := S_n(\mathcal{T}_n, \mathcal{F})$ be the associated finite-element space given in equation (1.8) and let $u_{I,n} \in S_n$ be the modified interpolant associated to $\mathcal{T}_n$ and $\mathcal{F}$, equation (2.10). Then, there exists $C > 0$, independent of n or u, such that*

$$\|u - u_{I,n}\|_{\mathcal{K}^1_{b+1}(\Omega)} \le C2^{-mn}\Big(\sum_j \|u_0\|_{\mathcal{K}^{m+1}_{a+b+1}(\Omega_j)} + \sum_Q |a_Q|\Big) =: C2^{-mn}\|u\|_V\,.$$

Note that while $u \in V := \hat{\mathcal{K}}^{m+1}_{a+b+1}(\Omega) \cap \mathcal{K}^1_{a+b+1}(\Omega) + U_s$, the difference $u - u_{I,n}$ satisfies $u - u_{I,n} \in \hat{\mathcal{K}}^{m+1}_{a+b+1}(\Omega) \cap \mathcal{K}^1_{a+b+1}(\Omega)$.

Proof. Again, we may assume that $w_s = 0$, by Lemma 2.23. The proof is then an immediate consequence of the estimates in Propositions 2.3 and 2.24 applied, respectively, to the triangles $T \in \mathcal{T}_0$ that have singular points as vertices and to the region U that is the complement of these triangles in Ω (as defined in the proof of Lemma 2.23 above), as follows:

$$\begin{aligned}\|u - u_{I,n}\|^2_{\mathcal{K}^1_{b+1}(\Omega)} &= \sum_{Q\in\mathcal{V}}\sum_{Q\in T} \|u - u_{I,n}\|^2_{\mathcal{K}^1_{b+1}(T)} + \|u - u_{I,n}\|^2_{\mathcal{K}^1_{b+1}(U)} \\ &\le C2^{-mn}\big(\sum_{Q\in\mathcal{V}}\sum_{Q\in T} \|u\|^2_{\mathcal{K}^{m+1}_{a+b+1}(T)} + \sum_{j=1}^{K} \|u\|^2_{\mathcal{K}^{m+1}_{a+b+1}(U\cap\Omega_j)}\big) \le C2^{-mn}\|u\|_V.\end{aligned}$$

The proof is now complete. □

3. Applications to the finite-element method

In this section, we apply the results of the previous sections to obtain quasi-optimal convergence rates for the finite-element solution of a transmission/interface problem, such as (0.1), using the meshes $\mathcal{T}_n$.

Recall that $\overline{\Omega} = \bigcup_{j=1}^K \overline{\Omega}_j$, where Ω_j are disjoint polygonal domains. Let $\Gamma := \partial\Omega \smallsetminus \bigcup_{j=1}^K \partial\Omega_k$ denote the interface. Assume the coefficients $A = [a_{ij}]$ have only jump discontinuities across the interface Γ. We are interested in approximating the solution of the boundary value/interface problem, (0.1), stated in the Introduction. Note that this problem is really formulated in a weak sense, which implies the usual matching and jump conditions at the interface,

$$u_+ = u_-, \quad D^A_{\nu+}u = D^A_{\nu-}u, \tag{3.1}$$

where we have labeled the nontangential limits u_+, u_- of u at each side of the interface, and denote the respective conormal derivatives, $D^A_{\nu+}$ and $D^A_{\nu-}$, by $D^A_{\nu\pm}u := \sum_{i,j}\nu_i A^{i,j}\partial_j u_\pm = \nu\cdot A\cdot\nabla u_\pm$, where ν is a choice of unit normal vector to the interface Γ. We shall also assume that $-\operatorname{div}(A\nabla)$ is *uniformly strongly elliptic*, in the usual sense, that is, we assume that there exists $C>0$ such that

$$\sum_{i,j=1}^{2} A^{i,j}(x)\xi_i\xi_j \geq C(\xi_1^2+\xi_2^2), \tag{3.2}$$

for all $x\in\Omega$ and $\xi\in\mathbb{R}^2$. Then, one of the main results of [37] states that the solution, u, of the boundary value/interface problem (0.1) is such that $u\in V$, where V is the approximation spaces introduced in equation (2.13) with $b=0$. Recall the norm $\|\ \|_V$ on the spaces $V := \hat{\mathcal{K}}^{m+1}_{a+1}(\Omega)\cap\mathcal{K}^1_{a+1}(\Omega)+U_s$ introduced in equation (2.14). Then, we have the following result.

Theorem 3.1. *Assume $\partial_D\Omega\neq\emptyset$ and that Ω is connected, then there exists $\eta>0$ with the following property. Assume $0<a<\eta$, $m\in\mathbb{Z}_+$, and $f\in\hat{\mathcal{K}}^{m-1}_{a-1}(\Omega)$, then there exists a unique solution $u=u_0+w_s$, $u_0\in\hat{\mathcal{K}}^{m+1}_{a+1}(\Omega)\cap\mathcal{K}^1_{a+1}(\Omega)$, $w_s\in U_s$, of equation* (0.1) *(the transmission problem). Moreover, there is a constant $C>0$ such that*

$$\|u\|_V := \|u_0\|_{\mathcal{K}^1_{a+1}(\Omega)} + \sum_{j=1}^{K}\|u_0\|_{\mathcal{K}^{m+1}_{a+1}(\Omega_j)} + \|w_s\|_{U_s} \leq C\sum_{j=1}^{K}\|f\|_{\mathcal{K}^{m-1}_{a-1}(\Omega_j)},$$

The constant C depends on Ω, m, a, and A^{ij}, but not on f.

In the case of the pure Neumann problem we have to take into account the nonuniqueness of the solution. Once we do that, a similar statement holds true.

The bilinear form associated to the problem (0.1) is $B(u,v)=\int_\Omega(\nabla u)\cdot A\cdot(\nabla v)dx$, as usual. Then, $u_n\in S_n=S(\mathcal{T}_n,\mathcal{F}_n)$ denotes the Galerkin approximation of the solution u of problem (0.1), namely, it is the unique $u_n\in S_n$ satisfying

$$B(u_n,v_n)=(f,v_n),\quad \forall\, v_n\in S_n. \tag{3.3}$$

We have the following theorem,

Theorem 3.2. *Let $m\geq 1$, assume the Dirichlet part of the boundary is not empty, and let u be the corresponding solution to problem* (0.1) *with $f\in\mathcal{K}^{m-1}_{a-1}(\Omega_j)$ for all j, where $0<a<\eta$, with η as in Theorem* 3.1. *Let $\mathcal{T}_n$ be the n-th κ-refinement of an initial triangulation $\mathcal{T}_0$ as in Definition* 2.7, *let $S_n := S_n(\mathcal{T}_n,m)$ be the associated finite-element space given in equation* (1.8) *and let $u_n=u_{S_n}\in S_n$ be the finite-element solution defined in* (3.3). *We assume $\kappa\leq\max\{2^{-m/a},1/2\}$. Then, there exists $C>0$, independent of f or n, such that*

$$\|u-u_n\|_{H^1(\Omega)} \leq C2^{-mn}\sum_{j=1}^{K}\|f\|_{\mathcal{K}^{m-1}_{a-1}(\Omega_j)}.$$

Proof. Notice first that since $a>0$, we have that $V\subset H^1(\Omega)$ continuously, that is, there exists $C>0$ such that $\|v\|_{H^1(\Omega)}\leq C\|v\|_V$, for all $v\in V$. Also, Céa's Lemma gives that there exists $C_1>0$ such that $\|u-u_n\|_{H^1(\Omega)}\leq C_1\|u-\tilde{u}_{I,n}\|_{H^1(\Omega)}$, where $u_{I,n}\in S_n=S(\mathcal{T}_n,\mathcal{F})$ is the modified interpolant associated to $\mathcal{T}_n$ and $\mathcal{F}$, equation

(2.10). Since $u \in V$, by Theorem 3.1 (quoted from [37]), an application of Theorem 2.25 and of Céa's Lemma give:

$$\|u - u_n\|_{H^1(\Omega)} \leq C\|u - u_{I,n}\|_{H^1(\Omega)} \leq C\|u - u_{I,n}\|_V \leq C2^{-mn}\|u\|_V \leq C2^{-mn}\sum_{j=1}^{K}\|f\|_{\mathcal{K}^{m-1}_{a-1}(\Omega_j)}.$$

The proof is now complete. □

A more convenient way of formulating the above theorem may be the following.

Theorem 3.3. *Under the notation and assumptions of Theorem* 3.2*,* $u_n \in S_n$ *satisfies*

$$\|u - u_n\|_{H^1(\Omega)} \leq C\dim(S_n)^{-m/d}\sum_j\|f\|_{\mathcal{K}^{m-1}_{a-1}(\Omega_j)},$$

for a constant $C > 0$ *independent of* f *and* n *(*$d = 2$*, the dimension of* Ω*).*

Proof. Again, let $\mathcal{T}_n$ be the triangulation of Ω after n refinements. Then, the number of triangles is $O(4^n)$ given the refinement procedure of Definition 2.6. Therefore, $\dim(S_n) \sim 4^n$ by Proposition 2.10, so that Theorem 2.25 gives

$$\|u - u_n\|_{H^1(\Omega)} \leq C2^{-nm}\sum_j\|f\|_{\mathcal{K}^{m-1}_{a-1}(\Omega_j)} \leq C\dim(S_n)^{-m/2}\sum_j\|f\|_{\mathcal{K}^{m-1}_{a-1}(\Omega_j)}.$$

This completes the proof. □

Using that $H^{m-1}(\Omega) \subset \mathcal{K}^{m-1}_{a-1}(\Omega)$ if $a \leq 1$, we obtain the following corollary.

Corollary 3.4. *Under the hypotheses of Theorem* 3.3*,*

$$\|u - u_n\|_{H^1(\Omega)} \leq C\dim(S_n)^{-m/2}\|f\|_{H^{m-1}(\Omega)},$$

for a constant $C > 0$ *independent of* $f \in H^{m-1}(\Omega)$ *and* n*.*

Note that we do not claim that $u \in \mathcal{K}^1_1(\Omega)$ (which is in general not true).

Finally, using a weighted Sobolev space duality argument, we are able to get an estimate of the error in the $L^2(\Omega)$ norm.

Theorem 3.5. *Under the notation and assumptions of Theorem* 2.25*,* $u_n \in S_n$ *satisfies*

$$\|u - u_n\|_{L^2(\Omega)} \leq C\dim(S_n)^{-(m+1)/2}\|f\|_{H^{m-1}(\Omega)},$$

for a constant $C > 0$ *independent of* f *and* n*.*

Proof. Consider the error equation for the bilinear form in (3.3),

$$B(\phi, v) = (u - u_n, v), \quad \forall\, v \in V.$$

Setting $v = u - u_n$ yields,

$$\|u - u_n\|^2_{L^2(\Omega)} = (u - u_n, u - u_n) = B(\phi, u - u_n).$$

Due to the orthogonality of the error in S_n, we know $B(u - u_n, v_n) = 0 \;\; \forall v_n \in S_n$. Thus,

$$\|u - u_n\|^2_{L^2(\Omega)} = B(\phi - \phi_{I,n}, u - u_n).$$

Using the Cauchy-Schwarz inequality gives,

$$\|u - u_n\|^2_{L^2(\Omega)} \leq \|u - u_n\|_{H^1(\Omega)}\|\phi - \phi_{I,n}\|_{H^1(\Omega)}.$$

Setting $m = 1$ and using the results from Theorem 2.25 (again setting $b = 0$) with the error equation yields,

$$\|\phi - \phi_{I,n}\|_{H^1(\Omega)} \leq \dim(S_n)^{-1/2} \|u - u_n\|_{L^2(\Omega)}.$$

Therefore, the proof is concluded by using Corollary 3.4 and some simplifications,

$$\|u - u_n\|^2_{L^2(\Omega)} \leq \Big(C \dim(S_n)^{-m/2} \|f\|_{H^{m-1}(\Omega)} \Big) \Big(\dim(S_n)^{-1/2} \|u - u_n\|_{L^2(\Omega)} \Big)$$

$$\Rightarrow \|u - u_n\|_{L^2(\Omega)} \leq C \dim(S_n)^{(-m-1)/2} \|f\|_{H^{m-1}(\Omega)}.$$

The proof is now complete. □

In addition to this, one can improve the regularity estimate of Theorem 3.3 as follows. Choose for each singular point, Q, a small neighborhood, Ω_Q. Assume that the sets, Ω_Q, are disjoint. Then, choose $0 < a_Q < \eta_Q$ such that $u|_{\Omega_Q} \in \mathcal{K}^{m+1}_{a_Q+1}(\Omega_Q) + U_s$ if $f \in \mathcal{K}^{m-1}_{a_Q-1}(\Omega_Q)$ for all Q and take $0 < \kappa_Q < 2^{-m/a_Q}$. For example, if for the Laplacian, Δ, and the same type of boundary conditions (both Dirichlet or both Neumann) on both sides of Q, take $a_Q < \pi/\alpha_Q$, where α_Q is the angle at Q. On the other hand, if at Q there are different types of boundary conditions, then take $a_Q < \pi/(2\alpha_Q)$. This allows the grading parameter to be controlled better and may lead to better meshes in practice. For instance, this restricts the need of "grading" to a few singular points.

3.1. **Textbook h^m-estimates.** Finally, one can obtain "textbook" h^m-error estimates as follows. Assume the function ρ is such that $0 \leq \rho \leq 1$ and that $\rho(x)$ is the distance to the singular point Q closest to x. Then, there exists an $\epsilon > 0$ such that, for any x such that $\rho(x) < \epsilon$, there will be a unique singular point Q closest to x. Consider a mesh $\mathcal{T}$ on Ω. For any triangle, T, in the given mesh, denote by d_T the diameter of the triangle and by $\rho_T = \inf_{x \in T} \rho(x)$, which is essentially the distance from T to the closest singular point of T. Let $a > 0$ be the constant arising in the regularity estimate of Theorem 3.2.

Then, assume that the mesh, $\mathcal{T}$, has the property that there exist constants $C_0 > 0$ and $\alpha > 0$ such that

- Any triangle T in the mesh that does *not* contain a singular point of Ω has angles $\geq \alpha$ and
$$d_T \leq C_0 h \rho_T^{1-a/m}.$$
- For any triangle T in the mesh that *does* contain a singular point of Ω, we have
$$d_T \leq C_0 h^{m/a}.$$

Denote by $u_{\mathcal{T}} \in S(\mathcal{T}, \mathcal{F})$ the finite-element solution associated to $\mathcal{T}$ and $\mathcal{F}$, then,

$$\|u - u_{\mathcal{T}}\|_{H^1(\Omega)} \leq C_1 h^m \sum_j \|f\|_{\mathcal{K}^{m-1}_{a-1}(\Omega_j)},$$

with a constant C_1 that depends only on C_0 and α. Provided that one constructs a mesh with "few" triangles, then the above estimate can be used to recover Theorem 3.3 by using estimates analogous to equations (2.11) and (2.8). See also [19], where similar conditions were provided.

Conclusion

We have shown that by using a sequence of graded meshes, optimal approximation results are obtained for functions in suitable L^p-based weighted Sobolev spaces. In this way, for $p = 2$, optimal approximation results are regained for the solution of mixed boundary value/transmission problems of type (0.1), whose solutions contain singularities. General conformally invariant families of finite elements with high order are considered in the context of these graded meshes and weighted spaces. Also, the approximation is in higher order weighted Sobolev spaces (not just of order one). Thus, for problems that require higher regularity or smoothness of the finite-element spaces, such as for problems of high order or those requiring higher order p-refinement, optimal results are still obtained. Future work involves extending these results to the least-squares finite-element method applied to problems with corner singularities [21, 26, 34, 35]. In these applications, the addition of the graded meshes can be used to show that the least-squares functional does in fact predict the optimal rate of convergence of the finite-element method. Another natural problem is to study the Multigrid method for the resulting spaces. See [16] for results in this direction.

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